

EXTENDED AFFINE ROOT SUPERSYSTEMS

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ABSTRACT. The interaction of a Lie algebra \mathcal{L} , having a weight space decomposition with respect to a nonzero toral subalgebra, with its corresponding root system forms a powerful tool in the study of the structure of \mathcal{L} . This, in particular, suggests a systematic study of the root system apart from its connection with the Lie algebra. Although there have been a lot of researches in this regard on Lie algebra level, such an approach has not been considered on Lie superalgebra level. In this work, we introduce and study extended affine root supersystems which are a generalization of both affine reflection systems and locally finite root supersystems. Extended affine root supersystems appear as the root systems of the super version of extended affine Lie algebras and invariant affine reflection algebras including affine Lie superalgebras. This work provides a framework to study the structure of this kind of Lie superalgebras referred to as extended affine Lie superalgebras.

0. INTRODUCTION

Lie algebras having a weight space decomposition with respect to a nonzero abelian subalgebra, called a toral subalgebra, form a vast class of Lie algebras. Locally finite split simple Lie algebras [11], extended affine Lie algebras [1], toral type extended affine Lie algebras [2], locally extended affine Lie algebras [10] and invariant affine reflection algebras [12] are examples of such Lie algebras. We can attach to such a Lie algebra, a subset of the dual space of its toral subalgebra called the root system. The interaction of such a Lie algebra with its root system offers an approach to study the structure of the Lie algebra via its root system. This in turn provokes a systematic study of the root system apart from its connection with the Lie algebra; see [1], [8], [17] and [12]. Although since 1977, when the concept of Lie superalgebras was introduced [6], there has been a significant number of researches on Lie superalgebras, the mentioned approach on Lie superalgebra level has not been considered in general. The first step towards such an approach is offering an abstract definition of the root system of a Lie superalgebra. In 1996, V. Serganova [15] introduced the notion of generalized root systems as a generalization of finite root systems; see also [4]. The main difference between generalized root systems and finite root systems is the existence of nonzero self-orthogonal roots. Serganova

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classified irreducible generalized root systems and showed that such root systems are root systems of finite dimensional basic classical simple Lie superalgebras [6] except for type $A(1, 1)$. She also gave two alternative definitions for generalized root systems. In a generalized root system for two self-orthogonal roots which are not orthogonal, either their summation or their subtraction (and not both) is again a root while according to the first alternative definition both summation and subtraction of two self-orthogonal roots which are not orthogonal, can be roots; this in particular allows to obtain type $A(1, 1)$ as well. In this work, we introduce extended affine root supersystems and systematically study them. Roughly speaking, a spanning set R of a nontrivial vector space over a field \mathbb{F} of characteristic zero, equipped with a symmetric bilinear form, is called an extended affine root supersystem if the root string property is satisfied. R is called a locally finite root supersystem if the form is nondegenerate. Irreducible locally finite root supersystems have been classified in [19]. One also knows from [19] that root string property for a locally finite root supersystem can be replaced by the locally finiteness of the real part. Generalized root systems according to the first alternative definition mentioned above, are nothing but finite locally finite root supersystems defined over the complex numbers. Locally finite root supersystems naturally appear in the theory of locally finite Lie superalgebras; see [13] and [20]. Extended affine root supersystems are extensions of locally finite root supersystems by abelian groups and appear as the root systems of extended affine Lie superalgebras introduced in [20]; in particular the root system of an affine Lie superalgebra [16] is an extended affine root supersystem. The nonzero elements of an extended affine root supersystem are divided into three disjoint parts: One consists of all real roots, i.e., the elements which are not self-orthogonal. The second part is the intersection of the radical of the form with the nonzero elements; the elements of this part are called isotropic roots. The last part consists of the elements which are not neither isotropic nor real and referred to as nonsingular roots. An extended affine root supersystem with no nonsingular root is called an affine reflection system [12] and an affine reflection system with no isotropic root is called a locally finite root system [8].

The concept of a base is so important in the theory of affine reflection systems and the corresponding Lie algebras. More precisely, reflectable bases are important in the study of the structure of locally extended affine root systems [18] and integral bases are important in the theory of locally finite Lie algebras [11]. A linearly independent subset Π of the set of real roots of an affine reflection system is called a reflectable base if all nonzero reduced real roots can be obtained from the iterated action of reflections based on the elements of Π . Reflectable bases for affine reflection systems have been studied in [3]. A linearly independent subset Π of a locally finite root supersystem R is called an integral base if each element of R can be written as a \mathbb{Z} -linear combination of the elements of Π .

In this work, we first derive some generic properties of extended affine root supersystems and locally finite root supersystems and then describe the structure of extended affine root supersystems. It is immediate from our results that an irreducible locally finite root supersystem can be recovered from a nonzero nonsingular root together with a reflectable base of the real part using the iterated action of reflections. We also show that each locally finite root supersystem R possesses an integral base and that if R is infinite, then it has an integral base Π with the property that each element of $R \setminus \{0\}$ can be written as $r_1\alpha_1 + \cdots + r_n\alpha_n$ in which

$r_1, \dots, r_n \in \{\pm 1\}$ and $\{\alpha_1, \dots, \alpha_n\} \subseteq \Pi$ with $r_1\alpha_1 + \dots + r_t\alpha_t \in R$ for all $1 \leq t \leq n$. The result of this paper forms a framework to study the locally finite basic classical simple Lie superalgebras [21].

1. GENERIC PROPERTIES

Throughout this work, \mathbb{F} is a field of characteristic zero. Unless otherwise mentioned, all vector spaces are considered over \mathbb{F} . We denote the dual space of a vector space V by V^* . We denote the degree of a homogenous element u of a superspace by $|u|$ and make a convention that if in an expression, we use $|u|$ for an element u of a superspace, by default we have assumed u is homogeneous. We denote the group of automorphisms of an abelian group A or a Lie superalgebra A by $\text{Aut}(A)$ and for a subset S of an abelian group, by $\langle S \rangle$, we mean the subgroup generated by S . For a set S , by $|S|$, we mean the cardinal number of S . For a map $f : A \rightarrow B$ and $C \subseteq A$, by $f|_C$, we mean the restriction of f to C . For two symbols i, j , by $\delta_{i,j}$, we mean the Kronecker delta, also $\dot{\cup}$ indicates the disjoint union. We finally recall that the direct union is, by definition, the direct limit of a direct system whose morphisms are inclusion maps.

In the sequel, by a *symmetric form* (with values in \mathbb{F}) on an additive abelian group A , we mean a map $(\cdot, \cdot) : A \times A \rightarrow \mathbb{F}$ satisfying

- $(a, b) = (b, a)$ for all $a, b \in A$,
- $(a + b, c) = (a, c) + (b, c)$ and $(a, b + c) = (a, b) + (a, c)$ for all $a, b, c \in A$.

In this case, we set $A^0 := \{a \in A \mid (a, A) = \{0\}\}$ and call it the *radical* of the form (\cdot, \cdot) . The form is called *nondegenerate* if $A^0 = \{0\}$. We note that if the form is nondegenerate, A is torsion free and we can identify A as a subset of $\mathbb{Q} \otimes_{\mathbb{Z}} A$. In the following, if an abelian group A is equipped with a nondegenerate symmetric form, we consider A as a subset of $\mathbb{Q} \otimes_{\mathbb{Z}} A$ without further explanation. Also if V is a vector space over a subfield \mathbb{K} of \mathbb{F} , by a *symmetric bilinear form* (with values in \mathbb{F}) on V , we mean a map $(\cdot, \cdot) : V \times V \rightarrow \mathbb{F}$ satisfying

- $(a, b) = (b, a)$; $(a, b \in V)$,
- $(ra + b, c) = r(a, c) + (b, c)$ and $(a, rb + c) = r(a, b) + (a, c)$; $(a, b, c \in V, r \in \mathbb{K})$.

We set $V^0 := \{a \in V \mid (a, V) = \{0\}\}$ and call it the *radical* of the form (\cdot, \cdot) . The form is called *nondegenerate* if $V^0 = \{0\}$. We draw the attention of the readers to the point that for a \mathbb{K} -vector space V equipped with a symmetric bilinear form (\cdot, \cdot) with values in \mathbb{F} and a subgroup A of V , the nondegeneracy of (\cdot, \cdot) and $(\cdot, \cdot)|_{A \times A}$ are not necessarily equivalent.

In this section, we first define an extended affine root supersystem and then study some generic properties of extended affine root supersystems. Proposition 1.11 and Lemma 1.17 are used to get the structure of extended affine root supersystems.

Definition 1.1. Suppose that A is a nontrivial additive abelian group, R is a subset of A and $(\cdot, \cdot) : A \times A \rightarrow \mathbb{F}$ is a symmetric form. Set

$$\begin{aligned} R^0 &:= R \cap A^0, \\ R^\times &:= R \setminus R^0, \\ R_{re}^\times &:= \{\alpha \in R \mid (\alpha, \alpha) \neq 0\}, \quad R_{re} := R_{re}^\times \cup \{0\}, \\ R_{ns}^\times &:= \{\alpha \in R \setminus R^0 \mid (\alpha, \alpha) = 0\}, \quad R_{ns} := R_{ns}^\times \cup \{0\}. \end{aligned}$$

We say $(A, (\cdot, \cdot), R)$ is an *extended affine root supersystem* if the following hold:

$$(S1) \quad 0 \in R \text{ and } \langle R \rangle = A,$$

$$(S2) \quad R = -R,$$

$$(S3) \quad \text{for } \alpha \in R_{re}^\times \text{ and } \beta \in R, 2(\alpha, \beta)/(\alpha, \alpha) \in \mathbb{Z},$$

$$(S4) \quad \begin{aligned} & \text{(root string property) for } \alpha \in R_{re}^\times \text{ and } \beta \in R, \text{ there are nonnegative integers } \\ & p, q \text{ with } 2(\beta, \alpha)/(\alpha, \alpha) = p - q \text{ such that} \\ & \{\beta + k\alpha \mid k \in \mathbb{Z}\} \cap R = \{\beta - p\alpha, \dots, \beta + q\alpha\}; \\ & \text{we call } \{\beta - p\alpha, \dots, \beta + q\alpha\} \text{ the } \alpha\text{-string through } \beta, \end{aligned}$$

$$(S5) \quad \text{for } \alpha \in R_{ns} \text{ and } \beta \in R \text{ with } (\alpha, \beta) \neq 0, \{\beta - \alpha, \beta + \alpha\} \cap R \neq \emptyset.$$

If there is no confusion, for the sake of simplicity, we say R is an *extended affine root supersystem* in A . Elements of R^0 are called *isotropic roots*, elements of R_{re} are called *real roots* and elements of R_{ns} are called *nonsingular roots*. A subset X of R^\times is called *connected* if each two elements $\alpha, \beta \in X$ are connected in X in the sense that there is a chain $\alpha_1, \dots, \alpha_n \in X$ with $\alpha_1 = \alpha$, $\alpha_n = \beta$ and $(\alpha_i, \alpha_{i+1}) \neq 0$, $i = 1, \dots, n-1$. We say an extended affine root supersystem R is *irreducible* if $R_{re} \neq \{0\}$ and R^\times is connected (equivalently, R^\times cannot be written as a disjoint union of two nonempty orthogonal subsets) and say it is *tame* if for each $\alpha \in R^0$, there is $\beta \in R^\times$ such that $\alpha + \beta \in R$. An extended affine root supersystem $(A, (\cdot, \cdot), R)$ is called a *locally finite root supersystem* if the form (\cdot, \cdot) is nondegenerate and it is called an *affine reflection system* if $R_{ns} = \{0\}$.

Example 1.2. (1) Suppose that \mathcal{L} is a finite dimensional basic classical simple Lie superalgebra with a Cartan subalgebra of the even part and corresponding root system R . One gets from the finite dimensional Lie superalgebra theory that R is a locally finite root supersystem; see [14].

(2) Suppose that \mathcal{L} is a contragredient Lie superalgebra of finite growth with symmetrizable Cartan matrix [16], then the corresponding root system is an extended affine root supersystem; see [20, Exa. 3.4 & Cor. 3.9].

Lemma 1.3. *Suppose that $(A, (\cdot, \cdot), R)$ is an extended affine root supersystem.*

(i) *If $\alpha \in R_{re}$ and $\delta \in R_{ns}$ with $(\delta, \alpha) \neq 0$, then there is a unique $r \in \{\pm 1\}$ such that $\delta + r\alpha \in R$.*

(ii) *If $\delta \in R_{ns}^\times$, then there is $\eta \in R_{ns}$ with $(\delta, \eta) \neq 0$.*

Proof. (i) By (S5), there is $r \in \{\pm 1\}$ such that $\delta + r\alpha \in R$. Suppose to the contrary that for r, s with $\{r, s\} = \{1, -1\}$, we have $\beta := \delta + s\alpha, \gamma := \delta + r\alpha \in R$. Since $(\beta, \delta), (\gamma, \delta) \neq 0$, we get $\beta, \gamma \notin R^0$. Also we know that at most one of the roots β, γ can be a nonsingular root. Suppose that β is a nonzero real root, then $(\beta, \beta) \neq 0$ and so $m := \frac{2s(\delta, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z} \setminus \{-1\}$. Since $\beta \in R_{re}^\times$, we have

$$\begin{aligned} \frac{m}{1+m} &= \frac{2s(\delta, \alpha)/(\alpha, \alpha)}{1 + 2s(\delta, \alpha)/(\alpha, \alpha)} = \frac{2s(\delta, \alpha)}{(\alpha, \alpha) + 2s(\delta, \alpha)} = \frac{2(\delta, \delta + s\alpha)}{(\delta + s\alpha, \delta + s\alpha)} \\ &= \frac{2(\delta, \beta)}{2(\beta, \beta)} \in \mathbb{Z}. \end{aligned}$$

This implies that $m = -2$. Now considering the $s\alpha$ -string through δ , we find nonnegative integers p, q with $p - q = -2$ such that $\{\delta + ks\alpha \mid k \in \mathbb{Z}\} \cap R =$

$\{\delta - ps\alpha, \dots, \delta + qs\alpha\}$; in particular as $\delta - s\alpha = \delta + r\alpha = \gamma \in R$, we have $\delta + 3s\alpha \in R$. But

$$(\delta + 3s\alpha, \delta + 3s\alpha) = 6s(\delta, \alpha) + 9(\alpha, \alpha) = -6(\alpha, \alpha) + 9(\alpha, \alpha) = 3(\alpha, \alpha) \neq 0$$

and

$$\frac{2(\alpha, \delta + 3s\alpha)}{(\delta + 3s\alpha, \delta + 3s\alpha)} = \frac{2(\alpha, \delta) + 6s(\alpha, \alpha)}{3(\alpha, \alpha)} = \frac{-2s(\alpha, \alpha) + 6s(\alpha, \alpha)}{3(\alpha, \alpha)} = \frac{4s}{3} \notin \mathbb{Z},$$

a contradiction. This completes the proof.

(ii) Since $\delta \in R_{ns}^\times$, we have $\delta \notin A^0$. Therefore, there is $\alpha \in R^\times$ with $(\delta, \alpha) \neq 0$. If α is nonsingular, we are done, so suppose $\alpha \in R_{re}^\times$. Set $n := \frac{2(\delta, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z}$. Considering the α -string through δ , we find nonnegative integers p, q with $p - q = n$ such that $\{k \in \mathbb{Z} \mid \delta + k\alpha \in R\} = \{-p, \dots, q\}$. Since $-p \leq -n \leq q$, we have $\eta := \delta - n\alpha \in R$. Now we have $(\delta, \eta) = (\delta, \delta - n\alpha) = -n(\delta, \alpha) \neq 0$ and $(\eta, \eta) = (\delta - n\alpha, \delta - n\alpha) = n^2(\alpha, \alpha) - 2n(\delta, \alpha) = 0$. So $\eta \in R_{ns}$ with $(\delta, \eta) \neq 0$. \square

Lemma 1.4. *Suppose that A is a nontrivial additive abelian group, R is a subset of A and $(\cdot, \cdot) : A \times A \rightarrow \mathbb{F}$ is a nondegenerate symmetric form. If $(A, (\cdot, \cdot), R)$ satisfies (S1), (S3) – (S5), then (S2) is also satisfied.*

Proof. We assume $\alpha \in R$. We must prove that $-\alpha \in R$. If $\alpha \in R_{re}^\times$, then the root string property implies that $\alpha - 2\alpha \in R$ and so $-\alpha \in R$. Next suppose that $\alpha \in R_{ns}^\times$, then using the same argument as in Lemma 1.3(ii), we find $\eta \in R_{ns}$ with $(\alpha, \eta) \neq 0$. So there is $r \in \{\pm 1\}$ with $\beta := \alpha + r\eta \in R$. Since $\beta \in R_{re}$, we have $-\beta \in R_{re}$. On the other hand, $(-\beta, \eta) \neq 0$, so either $-\beta + r\eta \in R$ or $-\beta - r\eta \in R$. But if $-\beta - r\eta = -\alpha - 2r\eta \in R$, we get $-\alpha - 2r\eta \in R_{re}^\times$ while

$$2 \frac{(\eta, -\beta - r\eta)}{(-\beta - r\eta, -\beta - r\eta)} = 2 \frac{(\eta, -\alpha - 2r\eta)}{(-\alpha - 2r\eta, -\alpha - 2r\eta)} = -r/2 \notin \mathbb{Z}$$

which is a contradiction. So $-\alpha = -\beta + r\eta \in R$. \square

Definition 1.5. Suppose that $(A, (\cdot, \cdot), R)$ is a locally finite root supersystem.

- The subgroup \mathcal{W} of $\text{Aut}(A)$ generated by r_α ($\alpha \in R_{re}^\times$) mapping $a \in A$ to $a - \frac{2(a, \alpha)}{(\alpha, \alpha)}\alpha$, is called the *Weyl group* of R .
- A subset S of R is called a *sub-supersystem* if the restriction of the form to $\langle S \rangle$ is nondegenerate, $0 \in S$, for $\alpha \in S \cap R_{re}^\times, \beta \in S$ and $\gamma \in S \cap R_{ns}$ with $(\beta, \gamma) \neq 0$, $r_\alpha(\beta) \in S$ and $\{\gamma - \beta, \gamma + \beta\} \cap S \neq \emptyset$.
- A sub-supersystem S of R is called *\mathbb{Z} -linearly closed* if $R \cap (\text{span}_{\mathbb{Z}} S) = S$.
- If $(A, (\cdot, \cdot), R)$ is irreducible, R is said to be of *real type* if $\text{span}_{\mathbb{Q}} R_{re} = \mathbb{Q} \otimes_{\mathbb{Z}} A$; otherwise, we say it is of *imaginary type*.
- If $\{R_i \mid i \in I\}$ is a class of sub-supersystems of R which are mutually orthogonal with respect the form (\cdot, \cdot) and $R \setminus \{0\} = \uplus_{i \in I} (R_i \setminus \{0\})$, we say R is the *direct sum* of R_i 's and write $R = \oplus_{i \in I} R_i$.
- The locally finite root supersystem $(A, (\cdot, \cdot), R)$ is called a *locally finite root system* if $R_{ns} = \{0\}$.
- $(A, (\cdot, \cdot), R)$ is said to be *isomorphic* to another locally finite root supersystem $(B, (\cdot, \cdot)', S)$ if there is a group isomorphism $\varphi : A \rightarrow B$ and a nonzero scalar $r \in \mathbb{F}$ such that $\varphi(R) = S$ and $(a_1, a_2) = r(\varphi(a_1), \varphi(a_2))'$ for all $a_1, a_2 \in A$.

Remark 1.6. (i) Locally finite root systems initially appeared in the work of K.H. Neeb and N. Stumme [11] on locally finite split simple Lie algebras. Then in 2003, O. Loos and E. Neher [8] systematically studied locally finite root systems. In their sense a locally finite root system is a locally finite spanning set R of a nontrivial vector space \mathcal{V} such that $0 \in R$ and for each $\alpha \in R \setminus \{0\}$, there is a functional $\check{\alpha} \in \mathcal{V}^*$ such that $\check{\alpha}(\alpha) = 2$, $\check{\alpha}(\beta) \in \mathbb{Z}$ for all $\beta \in R$ and that $\beta - \check{\alpha}(\beta)\alpha \in R$. It is proved that locally finiteness can be replaced by the existence of a nonzero bilinear form which is positive definite on the \mathbb{Q} -span of R and invariant under the Weyl group; moreover such a form is nondegenerate and is unique up to a scalar multiple if R is irreducible [8, §4.1]. Also a locally finite root system R in \mathcal{V} contains a \mathbb{Z} -basis for $\langle R \rangle$ [9, Lem. 5.1]. This allows us to have a natural isomorphism between \mathcal{V} and $\mathbb{F} \otimes_{\mathbb{Z}} \langle R \rangle$ and so it is natural to consider a locally finite root system as a subset of a torsion free abelian group instead of a subset of a vector space.

(ii) Suppose that S is a sub-supersystem of a locally finite root supersystem R , then S_{re} is a locally finite root system by [19, §3.1] and [8, §3.4]. Now the same argument as in [19, Lem. 3.12] shows that the root string property holds for S . This together with Lemma 1.4 implies that S is a locally finite root supersystem in its \mathbb{Z} -span.

Suppose that T is a nonempty index set with $|T| \geq 2$ and $\mathcal{U} := \oplus_{i \in T} \mathbb{Z}\epsilon_i$ is the free \mathbb{Z} -module over the set T . Define the form

$$\begin{aligned} (\cdot, \cdot) : \mathcal{U} \times \mathcal{U} &\longrightarrow \mathbb{F} \\ (\epsilon_i, \epsilon_j) &\mapsto \delta_{i,j}, \text{ for } i, j \in T, \end{aligned}$$

and set

$$\begin{aligned} \dot{A}_T &:= \{\epsilon_i - \epsilon_j \mid i, j \in T\}, \\ D_T &:= \dot{A}_T \cup \{\pm(\epsilon_i + \epsilon_j) \mid i, j \in T, i \neq j\}, \\ B_T &:= D_T \cup \{\pm\epsilon_i \mid i \in T\}, \\ C_T &:= D_T \cup \{\pm 2\epsilon_i \mid i \in T\}, \\ BC_T &:= B_T \cup C_T. \end{aligned} \tag{1.1}$$

These are irreducible locally finite root systems in their \mathbb{Z} -span's. Moreover, each irreducible locally finite root system is either an irreducible finite root system or a locally finite root system isomorphic to one of these locally finite root systems. We refer to locally finite root systems listed in (1.1) as *type A, D, B, C* and *BC* respectively. We note that if R is an irreducible locally finite root system as above, then $(\alpha, \alpha) \in \mathbb{N}$ for all $\alpha \in R$. This allows us to define

$$\begin{aligned} R_{sh} &:= \{\alpha \in R^\times \mid (\alpha, \alpha) \leq (\beta, \beta); \text{ for all } \beta \in R\}, \\ R_{ex} &:= R \cap 2R_{sh} \quad \text{and} \quad R_{lg} := R^\times \setminus (R_{sh} \cup R_{ex}). \end{aligned}$$

The elements of R_{sh} (resp. R_{lg}, R_{ex}) are called *short roots* (resp. *long roots*, *extra-long roots*) of R . We point out that following the usual notation in the literature, the locally finite root system of type *A* is denoted by \dot{A} instead of A , as all locally finite root systems listed above are spanning sets for $\mathbb{F} \otimes_{\mathbb{Z}} \mathcal{U}$ other than the one of type *A* which spans a subspace of codimension 1.

Lemma 1.7. (i) If $\{(X_i, (\cdot, \cdot)_i, S_i) \mid i \in I\}$ is a class of locally finite root supersystems, then for $X := \oplus_{i \in I} X_i$ and $(\cdot, \cdot) := \oplus_{i \in I} (\cdot, \cdot)_i$, $(X, (\cdot, \cdot), S := \cup_{i \in I} S_i)$ is a locally finite root supersystem.

(ii) Suppose that $(A, (\cdot, \cdot), R)$ is a locally finite root supersystem. Connectedness is an equivalence relation on $R \setminus \{0\}$. Also if S is a connected component of $R \setminus \{0\}$,

then $S \cup \{0\}$ is an irreducible sub-supersystem of R . Moreover, R is a direct sum of irreducible sub-supersystems.

(iii) Suppose that $(A, (\cdot, \cdot), R)$ is a locally finite root supersystem. For $A_{re} := \langle R_{re} \rangle$ and $(\cdot, \cdot)_{re} := (\cdot, \cdot)|_{A_{re} \times A_{re}}$, $(A_{re}, (\cdot, \cdot)_{re}, R_{re})$ is a locally finite root system.

Proof. See [19, §3]. \square

We also have the following straightforward lemma:

Lemma 1.8. Suppose that $(A, (\cdot, \cdot), R)$ is an irreducible locally finite root supersystem, set $\mathcal{V} := \mathbb{F} \otimes_{\mathbb{Z}} A$ and identify A as a subset of \mathcal{V} . Then $\mathcal{V} = \text{span}_{\mathbb{F}} R_{re}$ if and only if R is of real type.

In the following two theorems, we give the classification of irreducible locally finite root supersystems.

Theorem 1.9 ([19, Thm. 4.28]). Suppose that T, T' are index sets of cardinal numbers greater than 1 with $|T| \neq |T'|$ if T, T' are both finite. Fix a symbol α^* and pick $t_0 \in T$ and $p_0 \in T'$. Consider the free \mathbb{Z} -module $X := \mathbb{Z}\alpha^* \oplus \bigoplus_{t \in T} \mathbb{Z}\epsilon_t \oplus \bigoplus_{p \in T'} \mathbb{Z}\delta_p$ and define the symmetric form

$$(\cdot, \cdot) : X \times X \longrightarrow \mathbb{F}$$

by

$$\begin{aligned} (\alpha^*, \alpha^*) &:= 0, (\alpha^*, \epsilon_{t_0}) := 1, (\alpha^*, \delta_{p_0}) := 1 \\ (\alpha^*, \epsilon_t) &:= 0, (\alpha^*, \delta_q) := 0 & t \in T \setminus \{t_0\}, q \in T' \setminus \{p_0\} \\ (\epsilon_t, \delta_p) &:= 0, (\epsilon_t, \epsilon_s) := \delta_{t,s}, (\delta_p, \delta_q) := -\delta_{p,q} & t, s \in T, p, q \in T'. \end{aligned}$$

Take R to be $R_{re} \cup R_{ns}^\times$ as in the following table:

type	R_{re}	R_{ns}^\times
$A(0, T)$	$\{\epsilon_t - \epsilon_s \mid t, s \in T\}$	$\pm W\alpha^*$
$C(0, T)$	$\{\pm(\epsilon_t \pm \epsilon_s) \mid t, s \in T\}$	$\pm W\alpha^*$
$A(T, T')$	$\{\epsilon_t - \epsilon_s, \delta_p - \delta_q \mid t, s \in T, p, q \in T'\}$	$\pm W\alpha^*$

in which W is the subgroup of $\text{Aut}(X)$ generated by the reflections r_α ($\alpha \in R_{re} \setminus \{0\}$) mapping $\beta \in X$ to $\beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha$, then $(A := \langle R \rangle, (\cdot, \cdot)|_{A \times A}, R)$ is an irreducible locally finite root supersystem of imaginary type and conversely, each irreducible locally finite root supersystem of imaginary type is isomorphic to one and only one of these root supersystems.

Theorem 1.10 ([19, Thm. 4.37]). Suppose $n \in \{2, 3\}$ and $(X_1, (\cdot, \cdot)_1, S_1), \dots, (X_n, (\cdot, \cdot)_n, S_n)$, are irreducible locally finite root systems. Set $X := X_1 \oplus \dots \oplus X_n$ and $(\cdot, \cdot) := (\cdot, \cdot)_1 \oplus \dots \oplus (\cdot, \cdot)_n$ and consider the locally finite root system $(X, (\cdot, \cdot), S := S_1 \oplus \dots \oplus S_n)$. Take W to be the Weyl group of S . For $1 \leq i \leq n$, we identify X_i with a subset of $\mathbb{Q} \otimes_{\mathbb{Z}} X_i$ in the usual manner. If $1 \leq i \leq n$ and S_i is a finite root system of rank $\ell \geq 2$, we take $\{\omega_1^i, \dots, \omega_\ell^i\} \subseteq \mathbb{Q} \otimes_{\mathbb{Z}} X_i$ to be a set of fundamental weights for S_i (see [19, Pro. 2.7]) and if S_i is one of infinite locally finite root systems B_T, C_T, D_T or BC_T as in (1.1), by ω_1^i , we mean ϵ_1 , where 1 is a distinguished element of T . Also if S_i is one of the finite root systems $\{0, \pm\alpha\}$ of type A_1 or $\{0, \pm\alpha, \pm 2\alpha\}$ of type

BC_1 , we set $\omega_1^i := \frac{1}{2}\alpha$. Consider δ^* and $R := R_{re} \cup R_{ns}^\times$ as in the following table:

n	S_i ($1 \leq i \leq n$)	R_{re}	δ^*	R_{ns}^\times	type
2	$S_1 = A_\ell, S_2 = A_\ell$ ($\ell \in \mathbb{Z}^{\geq 1}$)	$S_1 \oplus S_2$	$\omega_1^1 + \omega_2^2$	$\pm \mathcal{W}\delta^*$	$A(\ell, \ell)$
2	$S_1 = B_T, S_2 = BC_{T'}$ ($ T , T' \geq 2$)	$S_1 \oplus S_2$	$\omega_1^1 + \omega_2^2$	$\mathcal{W}\delta^*$	$B(T, T')$
2	$S_1 = BC_T, S_2 = BC_{T'}$ ($ T , T' > 1$)	$S_1 \oplus S_2$	$\omega_1^1 + \omega_2^2$	$\mathcal{W}\delta^*$	$BC(T, T')$
2	$S_1 = BC_T, S_2 = BC_{T'}$ ($ T = 1, T' = 1$)	$S_1 \oplus S_2$	$2\omega_1^1 + 2\omega_2^2$	$\mathcal{W}\delta^*$	$BC(T, T')$
2	$S_1 = BC_T, S_2 = BC_{T'}$ ($ T = 1, T' > 1$)	$S_1 \oplus S_2$	$2\omega_1^1 + \omega_2^2$	$\mathcal{W}\delta^*$	$BC(T, T')$
2	$S_1 = D_T, S_2 = C_{T'}$ ($ T \geq 3, T' \geq 2$)	$S_1 \oplus S_2$	$\omega_1^1 + \omega_2^2$	$\mathcal{W}\delta^*$	$D(T, T')$
2	$S_1 = C_T, S_2 = C_{T'}$ ($ T , T' \geq 2$)	$S_1 \oplus S_2$	$\omega_1^1 + \omega_2^2$	$\mathcal{W}\delta^*$	$C(T, T')$
2	$S_1 = A_1, S_2 = BC_T$ ($ T = 1$)	$S_1 \oplus S_2$	$2\omega_1^1 + 2\omega_2^2$	$\mathcal{W}\delta^*$	$B(1, T)$
2	$S_1 = A_1, S_2 = BC_T$ ($ T \geq 2$)	$S_1 \oplus S_2$	$2\omega_1^1 + \omega_2^2$	$\mathcal{W}\delta^*$	$B(1, T)$
2	$S_1 = A_1, S_2 = C_T$ ($ T \geq 2$)	$S_1 \oplus S_2$	$\omega_1^1 + \omega_2^2$	$\mathcal{W}\delta^*$	$C(1, T)$
2	$S_1 = A_1, S_2 = B_3$	$S_1 \oplus S_2$	$\omega_1^1 + \omega_2^2$	$\mathcal{W}\delta^*$	$AB(1, 3)$
2	$S_1 = A_1, S_2 = D_T$ ($ T \geq 3$)	$S_1 \oplus S_2$	$\omega_1^1 + \omega_2^2$	$\mathcal{W}\delta^*$	$D(1, T)$
2	$S_1 = BC_1, S_2 = B_T$ ($ T \geq 2$)	$S_1 \oplus S_2$	$2\omega_1^1 + \omega_2^2$	$\mathcal{W}\delta^*$	$B(T, 1)$
2	$S_1 = BC_1, S_2 = G_2$	$S_1 \oplus S_2$	$2\omega_1^1 + \omega_2^2$	$\mathcal{W}\delta^*$	$G(1, 2)$
3	$S_1 = A_1, S_2 = A_1, S_3 = A_1$	$S_1 \oplus S_2 \oplus S_3$	$\omega_1^1 + \omega_2^2 + \omega_3^3$	$\mathcal{W}\delta^*$	$D(2, 1, \lambda)(\lambda \neq 0, -1)$
3	$S_1 = A_1, S_2 = A_1, S_3 := C_T$ ($ T \geq 2$)	$S_1 \oplus S_2 \oplus S_3$	$\omega_1^1 + \omega_2^2 + \omega_3^3$	$\mathcal{W}\delta^*$	$D(2, T)$

For $1 \leq i \leq n$, normalize the form $(\cdot, \cdot)_i$ on X_i such that $(\delta^*, \delta^*) = 0$ and that for type $D(2, T)$, $(\omega_1^1, \omega_1^1)_1 = (\omega_1^2, \omega_1^2)_2$. Then $(\langle R \rangle, (\cdot, \cdot) |_{\langle R \rangle \times \langle R \rangle}, R)$ is an irreducible locally finite root supersystem of real type and conversely, if $(X, (\cdot, \cdot), R)$ is an irreducible locally finite root supersystem of real type, it is either an irreducible locally finite root system or isomorphic to one of the locally finite root supersystems listed in the above table. Moreover, locally finite root supersystems in the above table are mutually non-isomorphic except for the ones of type $D(2, 1, \lambda)$. More precisely, For $\lambda, \mu \in \mathbb{F} \setminus \{0, -1\}$, $D(2, 1, \lambda)$ is isomorphic to $D(2, 1, \mu)$ if and only if λ, μ are in the same orbit under the action of the group of permutations on $\mathbb{F} \setminus \{0, -1\}$ generated by $\alpha \mapsto \alpha^{-1}$ and $\alpha \mapsto -1 - \alpha$.

We make a convention that from now on for the types listed in column “type” of Theorems 1.9 and 1.10, we may use a finite index set T and its cardinal number in place of each other, e.g., if T is a nonempty finite set of cardinal number ℓ , instead of type $B(1, T)$, we may write $B(1, \ell)$.

Proposition 1.11. *Suppose that $(A, (\cdot, \cdot), R)$ is an extended affine root supersystem and $\bar{\cdot} : A \longrightarrow \bar{A} := A/A^0$ is the canonical epimorphism. Suppose that $(\cdot, \cdot)^-$ is the induced form on \bar{A} defined by*

$$(\bar{a}, \bar{b}) := (a, b); \quad (a, b \in A).$$

Then we have the following:

- (i) $\{2(\alpha, \beta)/(\alpha, \alpha) \mid \alpha \in R_{re}^\times, \beta \in R\}$ is a bounded subset of \mathbb{Z} and for $\alpha \in R_{re}^\times$ and $\beta \in R_{ns}$, $2(\alpha, \beta)/(\alpha, \alpha) \in \{0, \pm 1, \pm 2\}$.
- (ii) If $\alpha, \beta \in R_{re}^\times$ are connected in R_{re} , then $(\alpha, \alpha)/(\beta, \beta) \in \mathbb{Q}$. Also each subset of R_{re}^\times whose elements are mutually disconnected in R_{re}^\times is \mathbb{Z} -linearly independent.
- (iii) $(\bar{A}, (\cdot, \cdot)^-, \bar{R})$ is a locally finite root supersystem. Moreover, if R is irreducible, then so is \bar{R} .

Proof. (i) See [19, Lem. 3.7] and follow the proof of [19, Lem. 3.8].

(ii) See [19, Lem. 3.6].

(iii) Set $\mathcal{V} := \mathbb{F} \otimes_{\mathbb{Z}} \bar{A}$. Since \bar{A} is torsion free, we identify \bar{A} as a subset of \mathcal{V} and set $\mathcal{V}_{\mathbb{Q}} := \text{span}_{\mathbb{Q}} \bar{R}$ as well as $\mathcal{V}_{re} := \text{span}_{\mathbb{Q}} \bar{R}_{re}$. The nondegenerate form $(\cdot, \cdot)^- : \bar{A} \times \bar{A} \longrightarrow \mathbb{F}$ induces a bilinear form

$$(\cdot, \cdot)_{\mathbb{F}} : (\mathbb{F} \otimes_{\mathbb{Z}} \bar{A}) \times (\mathbb{F} \otimes_{\mathbb{Z}} \bar{A}) \longrightarrow \mathbb{F}$$

$$(r \otimes \bar{a}, s \otimes \bar{b}) := rs(a, b); \quad (r, s \in \mathbb{F}, a, b \in A).$$

Take $(\cdot, \cdot)_{\mathbb{Q}}$ to be the restriction of the form $(\cdot, \cdot)_{\mathbb{F}}$ to $\mathcal{V}_{\mathbb{Q}} = \text{span}_{\mathbb{Q}} \bar{R}$. Using the same argument as in [3, Lem. 1.6], one can see that $(\cdot, \cdot)_{\mathbb{Q}}$ is nondegenerate. To carry out the proof, we just need to verify the root string property. To this end using [19, Lem.'s 3.10 & 3.12], it is enough to show that $\bar{R}_{re} = \overline{R_{re}} = \{\bar{\alpha} \mid \alpha \in R_{re}\} \subseteq \mathbb{F} \otimes \bar{A}$ is locally finite in $\mathcal{V}_{re} = \text{span}_{\mathbb{Q}} \bar{R}_{re}$ in the sense that it intersects each finite dimensional subspace of \mathcal{V}_{re} in a finite set. Now we assume \mathcal{W} is a finite dimensional subspace of \mathcal{V}_{re} and show that $\bar{R}_{re} \cap \mathcal{W}$ is a finite set. Since \mathcal{W} is a finite dimensional subspace of \mathcal{V}_{re} , there is a finite subset $\{\alpha_1, \dots, \alpha_m\} \subseteq R_{re}$ such that $\mathcal{W} \subseteq U_1 := \text{span}_{\mathbb{Q}}\{\bar{\alpha}_1, \dots, \bar{\alpha}_m\}$. By [19, Lem. 3.1], there is a finite dimensional subspace U_2 of $\mathcal{V}_{\mathbb{Q}}$ such that $U_1 \subseteq U_2$ and the form $(\cdot, \cdot)_{\mathbb{Q}}$ restricted to U_2 is nondegenerate. Suppose that $\{R_i \mid i \in I\}$ is the class of connected components of R_{re}^{\times} . To complete the proof using part (ii) together with the fact that U_2 is finite dimensional, we need to show that for all $i \in I$, $U_2 \cap \bar{R}_i$ is a finite set. Since U_2 is finite dimensional, there is a finite set $\{\beta_1, \dots, \beta_n\} \subseteq R$ such that $U_2 \subseteq \text{span}_{\mathbb{Q}}\{\bar{\beta}_1, \dots, \bar{\beta}_n\}$. Fix $i \in I$ and consider the map

$$\begin{aligned} \varphi : U_2 \cap \bar{R}_i &\longrightarrow \mathbb{Z}^n \\ \bar{\alpha} &\mapsto \left(\frac{2(\bar{\alpha}, \bar{\beta}_1)}{(\bar{\alpha}, \bar{\alpha})}, \dots, \frac{2(\bar{\alpha}, \bar{\beta}_n)}{(\bar{\alpha}, \bar{\alpha})} \right). \end{aligned}$$

We claim that φ is one to one. Suppose that for $\alpha, \beta \in R_i$, $\bar{\alpha}, \bar{\beta} \in U_2 \cap \bar{R}_i$ and

$$\left(\frac{2(\bar{\alpha}, \bar{\beta}_1)}{(\bar{\alpha}, \bar{\alpha})}, \dots, \frac{2(\bar{\alpha}, \bar{\beta}_n)}{(\bar{\alpha}, \bar{\alpha})} \right) = \left(\frac{2(\bar{\beta}, \bar{\beta}_1)}{(\bar{\beta}, \bar{\beta})}, \dots, \frac{2(\bar{\beta}, \bar{\beta}_n)}{(\bar{\beta}, \bar{\beta})} \right),$$

then for $1 \leq i \leq n$, $\frac{(\bar{\alpha}, \bar{\beta}_i)}{(\bar{\alpha}, \bar{\alpha})} = \frac{(\bar{\beta}, \bar{\beta}_i)}{(\bar{\beta}, \bar{\beta})}$. So $(\frac{\bar{\alpha}}{(\bar{\alpha}, \bar{\alpha})} - \frac{\bar{\beta}}{(\bar{\beta}, \bar{\beta})}, \bar{\beta}_i)_{\mathbb{F}} = 0$ for all $1 \leq i \leq n$.

Therefore, $(\frac{\bar{\alpha}}{(\bar{\alpha}, \bar{\alpha})} - \frac{\bar{\beta}}{(\bar{\beta}, \bar{\beta})}, U_2)_{\mathbb{F}} = \{0\}$. But $\frac{(\bar{\alpha}, \bar{\alpha})}{(\bar{\beta}, \bar{\beta})} \in \mathbb{Q}$ (see part (ii)) and so

$$(\bar{\alpha} - \frac{(\bar{\alpha}, \bar{\alpha})}{(\bar{\beta}, \bar{\beta})} \bar{\beta}, U_2)_{\mathbb{Q}} = (\bar{\alpha} - \frac{(\bar{\alpha}, \bar{\alpha})}{(\bar{\beta}, \bar{\beta})} \bar{\beta}, U_2)_{\mathbb{F}} = \{0\}.$$

So we get that $\bar{\alpha} = \frac{(\bar{\alpha}, \bar{\alpha})}{(\bar{\beta}, \bar{\beta})} \bar{\beta}$ as the form $(\cdot, \cdot)_{\mathbb{Q}}$ on U_2 is nondegenerate. But as $\frac{2(\bar{\alpha}, \bar{\beta})}{(\bar{\alpha}, \bar{\alpha})}, \frac{2(\bar{\alpha}, \bar{\beta})}{(\bar{\beta}, \bar{\beta})} \in \mathbb{Z}$, we get that $(\bar{\alpha}, \bar{\alpha})/(\bar{\beta}, \bar{\beta}) \in \{\pm 1, \pm 2, \pm \frac{1}{2}\}$. If $\frac{(\bar{\alpha}, \bar{\alpha})}{(\bar{\beta}, \bar{\beta})} = \pm 2$, then $\bar{\alpha} = \pm 2\bar{\beta}$ and so $\frac{(\bar{\alpha}, \bar{\alpha})}{(\bar{\beta}, \bar{\beta})} = 4$, a contradiction, also if $\frac{(\bar{\alpha}, \bar{\alpha})}{(\bar{\beta}, \bar{\beta})} = \pm(1/2)$, then $\bar{\alpha} = \pm(1/2)\bar{\beta}$ and $(\bar{\alpha}, \bar{\alpha})/(\bar{\beta}, \bar{\beta}) = 1/4$ which is again a contradiction. If $(\bar{\alpha}, \bar{\alpha}) = -(\bar{\beta}, \bar{\beta})$, then $\bar{\alpha} = -\bar{\beta}$ and so $(\bar{\alpha}, \bar{\alpha})/(\bar{\beta}, \bar{\beta}) = 1$ that is absurd. Therefore, $\bar{\alpha} = \bar{\beta}$ i.e., φ is one to one. Also using part (i), we get that the set $\{\frac{2(\bar{\alpha}, \bar{\beta})}{(\bar{\alpha}, \bar{\alpha})} \mid \alpha \in R_{re}, \beta \in R\}$ is bounded. This in turn implies that the image of φ and so $U_2 \cap \bar{R}_i$ is finite. This together with Lemma 1.7 completes the proof of the first assertion. The last assertion follows from an immediate verification. \square

Definition 1.12. Suppose that $(A, (\cdot, \cdot), R)$ is an irreducible extended affine root supersystem. We define the *type* of R to be the type of \bar{R} .

Lemma 1.13. Suppose that A is a torsion free abelian group and $(A, (\cdot, \cdot), R)$ is an irreducible extended affine root supersystem of type $X \neq A(\ell, \ell), BC(1, 1)$. Then for each $a \in A^0$, there is a nonzero integer n such that $na \in \langle R^0 \rangle$; in particular, if $X \neq A(\ell, \ell)$, $R^0 = \{0\}$ if and only if $A^0 = \{0\}$.

Proof. Set $\mathcal{V} := \mathbb{Q} \otimes_{\mathbb{Z}} A$. Since A is torsion free, we identify A as a subset of \mathcal{V} . The form (\cdot, \cdot) induces the symmetric bilinear form $\mathcal{V} \times \mathcal{V} \longrightarrow \mathbb{F}$ (with values in \mathbb{F}) defined by $(r \otimes a, s \otimes b) := rs(a, b)$ ($r, s \in \mathbb{Q}, a, b \in A$); we denote this bilinear form

again by (\cdot, \cdot) . Set $\mathcal{V}^0 := \{\alpha \in \mathcal{V} \mid (\alpha, \mathcal{V}) = \{0\}\}$. Suppose that $\bar{\cdot} : \mathcal{V} \longrightarrow \bar{\mathcal{V}} := \mathcal{V}/\mathcal{V}^0$ is the canonical epimorphism and that $(\bar{\cdot}, \bar{\cdot})$ is the induced map on $\bar{\mathcal{V}} \times \bar{\mathcal{V}}$. We note that $\mathcal{V}^0 = \text{span}_{\mathbb{Q}} A^0$ and use Proposition 1.11 and Lemma 1.7 to get that \bar{R}_{re} is a locally finite root system in its \mathbb{Z} -span. Therefore by [9, Lem. 5.1], there is a \mathbb{Z} -basis $B \subseteq \bar{R}_{re}$ for $\bar{A}_{re} := \langle \bar{R}_{re} \rangle$ such that

$$(1.2) \quad \mathcal{W}_B B = (\bar{R}_{re})_{red}^\times := \bar{R}_{re} \setminus \{2\bar{\alpha} \mid \alpha \in R_{re}\},$$

in which by \mathcal{W}_B , we mean the subgroup of the Weyl group of \bar{R}_{re} generated by $r_{\bar{\alpha}}$ for all $\bar{\alpha} \in B$. Fix $\alpha^* \in \bar{R}_{ns}^\times$ if \bar{R} is of imaginary type and set

$$K := \begin{cases} B & \text{if } \bar{R} \text{ is of real type,} \\ B \cup \{\alpha^*\} & \text{if } \bar{R} \text{ is of imaginary type.} \end{cases}$$

Then K is a basis for \mathbb{Q} -vector space $\bar{\mathcal{V}}$. Take $\dot{K} \subseteq R$ to be a preimage of K under the canonical map “ $\bar{\cdot}$ ”, then \dot{K} is a \mathbb{Q} -linearly independent subspace of \mathcal{V} and for $\dot{\mathcal{V}} := \text{span}_{\mathbb{Q}} \dot{K}$, we have $\mathcal{V} = \dot{\mathcal{V}} \oplus \mathcal{V}^0$. Now set $\dot{R} := \{\dot{\alpha} \in \dot{\mathcal{V}} \mid \exists \sigma \in \mathcal{V}^0; \dot{\alpha} + \sigma \in R\}$ and for each $\dot{\alpha} \in \dot{R}$, set $T_{\dot{\alpha}} := \{\sigma \in \mathcal{V}^0 \mid \dot{\alpha} + \sigma \in R\}$. Then \dot{R} is a locally finite root supersystem in its \mathbb{Z} -span isomorphic to \bar{R} . Since $\dot{K} \subseteq R \cap \dot{R}$, we have $-\dot{K} \subseteq R \cap \dot{R}$. Taking $\mathcal{W}_{\dot{K}}$ to be the subgroup of the Weyl group of R generated by the reflections based on real roots of \dot{K} , we have

$$\mathcal{W}_{\dot{K}}(\pm \dot{K}) \subseteq R \cap \dot{R} \quad \text{and} \quad \pm \mathcal{W}_{\dot{K}} \dot{K} = \begin{cases} (\dot{R}_{re})_{red}^\times & \text{if } \bar{R} \text{ is of real type,} \\ \dot{R}^\times & \text{if } \bar{R} \text{ is of imaginary type.} \end{cases}$$

So

$$(1.3) \quad \begin{cases} 0 \in T_{\dot{\alpha}} & \text{if } \dot{R} \text{ is of real type and } \dot{\alpha} \in (\dot{R}_{re})_{red} := (\dot{R}_{re})_{red} \cup \{0\}, \\ 0 \in T_{\dot{\alpha}} & \text{if } \dot{R} \text{ is of imaginary type and } \dot{\alpha} \in \dot{R}. \end{cases}$$

To proceed with the proof, we claim that for each $\dot{\alpha} \in \dot{R}$ and $\sigma \in T_{\dot{\alpha}}$, there is $n \in \mathbb{Z} \setminus \{0\}$ such that $n\sigma \in \langle R^0 \rangle$. If $\dot{\alpha} = 0$, $T_{\dot{\alpha}} \subseteq R^0$ and there is nothing to prove. Now the following cases can happen:

Case 1. $\dot{\alpha} \in \dot{R}_{re}^\times$: In this case, we show that $T_{\dot{\alpha}} \subseteq R^0$. We first assume $\dot{\alpha} \in (\dot{R}_{re})_{red}^\times$, then since $0 \in T_{\dot{\alpha}}$, $\alpha := \dot{\alpha}$, $\beta := \dot{\alpha} + \sigma \in R$. Now considering the α -string through β , we find that $\sigma \in R$ and so it is an element of R^0 . Next suppose that $\dot{\alpha} \in \dot{R}_{re}^\times \setminus (\dot{R}_{re})_{red}$, then there exists $\dot{\beta} \in (\dot{R}_{re})_{red}$ with $\dot{\alpha} = 2\dot{\beta}$. Now for $\sigma \in T_{\dot{\alpha}}$, taking $\alpha := \dot{\beta}$ and $\beta := \dot{\alpha} + \sigma$ and considering the α -string through β , we get that $\sigma \in R^0$.

Case 2. \dot{R} is of real type and $\dot{\alpha} \in \dot{R}_{ns}^\times$: For $\dot{\gamma} \in (\dot{R}_{re})_{red}^\times$ and $\eta \in T_{\dot{\alpha}}$, since $\dot{\gamma} \in R_{re}^\times$, we have $r_{\dot{\gamma}}(\dot{\alpha} + \eta) = r_{\dot{\gamma}}(\dot{\alpha}) + \eta \in R$. This implies that $T_{\dot{\alpha}} \subseteq T_{r_{\dot{\gamma}}(\dot{\alpha})}$; similarly we have $T_{r_{\dot{\gamma}}(\dot{\alpha})} \subseteq T_{\dot{\alpha}}$. We know that the Weyl group $\dot{\mathcal{W}}$ of \dot{R} is generated by the reflections based on nonzero elements of $(\dot{R}_{re})_{red}$ and that each two nonzero nonsingular roots are $\dot{\mathcal{W}}$ -conjugate as \dot{R} is not of type $A(\ell, \ell)$. These altogether imply that $T := T_{\dot{\alpha}} = T_{\dot{\beta}}$ for all nonzero nonsingular roots $\dot{\beta}$. Since \dot{R} is of real type $X \neq BC(1, 1), A(\ell, \ell)$, one finds nonsingular roots $\dot{\beta}, \dot{\gamma}$ with $(\dot{\gamma}, \dot{\beta}) \neq 0$, $\dot{\beta} - \dot{\gamma} \in \dot{R}_{re}$ and $\dot{\beta} + \dot{\gamma} \notin \dot{R}$. We next note that $T = T_{\dot{\alpha}} = -T_{-\dot{\alpha}} = -T$ and fix $\sigma, \tau \in T = -T$. Since $\alpha := \dot{\beta} + \sigma, \beta := \dot{\gamma} + \tau, \gamma := \dot{\gamma} - \tau \in R$ and $(\alpha, \beta), (\alpha, \gamma) \neq 0$, there are $r, s \in \{\pm 1\}$ with $\zeta := \alpha + r\beta, \eta := \alpha + s\gamma \in R$. But $\dot{\beta} + \dot{\gamma} \notin \dot{R}$, so $\zeta = \dot{\beta} - \dot{\gamma} + \sigma - \tau, \eta = \dot{\beta} - \dot{\gamma} + \sigma + \tau$. Therefore using the previous case, we have $\sigma - \tau, \sigma + \tau \in R^0$; this in particular implies that $2\sigma, 2\tau \in \langle R^0 \rangle$.

Case 3. \dot{R} is of imaginary type and $\dot{\alpha} \in \dot{R}_{ns}^\times$: By [19, Lem. 4.5], there is $\dot{\beta} \in \dot{R}_{re}$ such that $(\dot{\alpha}, \dot{\beta}) \neq 0$. We next note that $T := T_{\dot{\alpha}} = -T_{-\dot{\alpha}}$. Also as $0 \in T_{\dot{\beta}}$ and R is invariant under the reflections, $T := T_{\dot{\alpha}} = T_{r_{\dot{\beta}}(\dot{\alpha})}$ as in the previous case. Also by [19, Lem.'s 4.6 & 4.7], we have $r_{\dot{\beta}}\dot{\alpha} - \dot{\alpha} \in \dot{R}_{re}$ while $r_{\dot{\beta}}\dot{\alpha} + \dot{\alpha} \notin \dot{R}$. Now for $\sigma, \tau \in T$, we have $(r_{\dot{\beta}}\dot{\alpha} + \sigma, \dot{\alpha} + \tau) \neq 0$. Since $r_{\dot{\beta}}\dot{\alpha} + \dot{\alpha} \notin \dot{R}$, we get that $r_{\dot{\beta}}\dot{\alpha} - \dot{\alpha} + \sigma - \tau \in R_{re}$ and so using Case 1, we have $\sigma - \tau \in R^0$. Thus we have $T - T \subseteq R^0$; but $0 \in T$, so $T = T_{\dot{\alpha}} \subseteq R^0$.

Now suppose $a \in A^0 \setminus \{0\}$, then $a \in \mathcal{V}^0$ and there are $r_1, \dots, r_m \in \mathbb{Z} \setminus \{0\}$ and $\alpha_1, \dots, \alpha_m \in R \setminus \{0\}$ with $a = \sum_{i=1}^m r_i \alpha_i$. But for each $1 \leq i \leq m$, there are $\dot{\alpha}_i \in \dot{R}$, $n_i \in \mathbb{Z} \setminus \{0\}$ and $\delta_i \in \langle R^0 \rangle$ with $\alpha_i = \dot{\alpha}_i + \frac{1}{n_i} \delta_i$, so $a = \sum_{i=1}^m r_i \dot{\alpha}_i + \sum_{i=1}^m \frac{r_i}{n_i} \delta_i$. This implies that $a = \sum_{i=1}^m \frac{r_i}{n_i} \delta_i$. Therefore we have $n_1 \cdots n_m a \in \langle R^0 \rangle$.

For the last assertion, we just need to assume R is of type $BC(1, 1)$. In this case, regarding the description $R = \cup_{\dot{\alpha} \in \dot{R}} (\dot{\alpha} + T_{\dot{\alpha}})$ for R as above, $T_{\dot{\alpha}} \subseteq R^0$ for $\dot{\alpha} \in \dot{R}_{re}$ as in Case 1. Now suppose $R^0 = \{0\}$, so $T_{\dot{\alpha}} = \{0\}$ for $\dot{\alpha} \in \dot{R}_{re}$. Suppose that $\dot{R} = \{0, \pm \epsilon_0, \pm \delta_0, \pm 2\epsilon_0, \pm 2\delta_0, \pm \epsilon_0 \pm \delta_0\}$. Now if $r, s \in \{\pm 1\}$ and $\delta \in T_{r\epsilon_0 + s\delta_0}$, since $(r\epsilon_0, r\epsilon_0 + s\delta_0 + \delta) \neq 0$, we get that $s\delta_0 + \delta \in R$ and so $\delta \in T_{s\delta_0} = \{0\}$. This shows that $R \subseteq \dot{R}$ and so $\mathcal{V}^0 = \{0\}$ which in turn implies that $A^0 = \{0\}$. This completes the proof. \square

The following example shows that the condition $X \neq A(\ell, \ell)$ is necessary in Lemma 1.13. This is a phenomena occurring in the super-version of root systems; more precisely, one knows that for an affine reflection system $(A, (\cdot, \cdot), R)$ i.e., an extended affine root supersystem with no nonsingular root, $R^0 = \{0\}$ if and only if $A^0 = \{0\}$; see [3].

Example 1.14. (i) Suppose that $(\dot{A}, (\cdot, \cdot), \dot{R})$ is a locally finite root supersystem of type $X = A(\ell, \ell)$ for some integer $\ell \geq 2$ as in Theorem 1.10 with Weyl group \mathcal{W} . Suppose that σ is a symbol and set $A := \dot{A} \oplus \mathbb{Z}\sigma$. Fix $\delta^* \in \dot{R}_{ns}^\times$ and note that $-\delta^* \notin \mathcal{W}\delta^*$. Set $R := \dot{R}_{re} \cup \pm(\mathcal{W}\delta^* + \sigma)$. Extend the form on \dot{A} to a form on A denoted again by (\cdot, \cdot) such that σ is an element of the radical of this new form. Set $B := \langle R \rangle$. We claim that the form (\cdot, \cdot) restricted to B is degenerate; indeed, since \dot{R} is of real type, there is $n \in \mathbb{Z} \setminus \{0\}$ such that $n\delta^* \in \langle \dot{R}_{re} \rangle \subseteq B$, so $n\sigma = n(\delta^* + \sigma) - n\delta^* \in B$ which in turn implies that $n\sigma$ is an element of the radical of the form on B . One can check that for $\alpha \in R_{ns}$ and $\beta \in R$ with $(\alpha, \beta) \neq 0$, we have either $\alpha + \beta \in R$ or $\alpha - \beta \in R$. Next we note that $R^0 = \{0\}$, the root string property is satisfied for \dot{R}_{re} and that for $\alpha \in R_{re}^\times$ and $\beta \in R$, we have $r_\alpha \beta \in R$. These together with the same argument as in [19, Lem. 3.12] imply that the root string property is satisfied for R . These all together imply that R is an extended affine root supersystem with $R^0 = \{0\}$ but it is not a locally finite root supersystem as the form on B is degenerate.

(ii) Suppose that $(\dot{A}, (\cdot, \cdot), \dot{R})$ is a locally finite root supersystem of type $A(1, 1)$ as in Theorem 1.10. Suppose that σ is a symbol and set $A := \dot{A} \oplus \mathbb{Z}\sigma$. Set $R := \dot{R}_{re} \cup (\dot{R}_{ns}^\times \pm \sigma)$. Extend the form on \dot{A} to a form on A denoted again by (\cdot, \cdot) such that σ is an element of the radical of this new form. As above, the form restricted to $B := \langle R \rangle$ is degenerate and R is an extended affine root supersystem, with $R^0 = \{0\}$, which is not a locally finite root supersystem.

Lemma 1.15. *Suppose that $(A, (\cdot, \cdot), R)$ is a locally finite root supersystem. Then we have the following:*

(i) There is a sub-supersystem S of R with $R_{ns} = S_{ns}$ and $\langle R \rangle = \langle S \rangle$ such that for $\alpha \in S$ and $\delta \in S_{ns}$ with $(\alpha, \delta) \neq 0$, there is a unique $r \in \{\pm 1\}$ such that $\alpha + r\delta \in S$.

(ii) Identify A as a subset of $\mathbb{F} \otimes_{\mathbb{Z}} A$. If $\delta \in R_{ns}^\times$ and $k \in \mathbb{F}$ with $k\delta \in R$, then $k \in \{0, \pm 1\}$.

Proof. (i) Without loss of generality, we assume R is irreducible. If R is an irreducible locally finite root supersystem of type $X \neq A(1, 1), BC(T, T'), C(T, T')$ ($|T|, |T'| \geq 1$), we take $R = S$. Next suppose R is of type $X = A(1, 1), BC(T, T'), C(T, T')$. We know that $R_{re} = R^1 \oplus R^2$ with R^1, R^2 as following:

X	R^1	R^2
$A(1, 1)$	$\{0, \pm\alpha\}$	$\{0, \pm\beta\}$
$BC(T, T')$	$\{\pm\epsilon_i, \pm\epsilon_i \pm \epsilon_j \mid i, j \in T\}$	$\{\pm\delta_p, \pm\delta_p \pm \delta_q \mid p, q \in T'\}$
$C(1, T)$	$\{0, \pm\alpha\}$	$\{\pm\epsilon_i \pm \epsilon_j \mid i, j \in T\}$
$C(T, T')$	$\{\pm\epsilon_i \pm \epsilon_j \mid i, j \in T\}$	$\{\pm\delta_p \pm \delta_q \mid p, q \in T'\}$

Now take $S = R_{ns} \cup S^1 \cup S^2$ where S^1, S^2 are considered as in the following table:

X	S^1	S^2
$A(1, 1)$	$\{0, \pm\alpha\}$	$\{0\}$
$BC(T, T')$	$\{0, \pm\epsilon_i, \pm\epsilon_i \pm \epsilon_j \mid i, j \in T, i \neq j\}$	R^2
$C(1, T)$	$\{0\}$	R^2
$C(T, T')$	$\{0, \pm\epsilon_i \pm \epsilon_j \mid i, j \in T, i \neq j\}$	R^2

This completes the proof.

(ii) As in the proof of Proposition 1.11, the form (\cdot, \cdot) on A induces an \mathbb{F} -bilinear form on $\mathbb{F} \otimes_{\mathbb{Z}} A$ which is denoted by $(\cdot, \cdot)_{\mathbb{F}}$ and satisfies $(r \otimes a, s \otimes b)_{\mathbb{F}} = rs(a, b)$ for $r, s \in \mathbb{F}$ and $a, b \in A$. Now suppose $\delta \in R_{ns}^\times$ and $k \neq 0$ with $k\delta \in R$. Since $\delta \in R_{ns}^\times$, there is $\beta \in R$ with $(\delta, \beta) \neq 0$. Then for $\beta' := r_\beta(\delta)$, we have $(\beta', \beta') = 0$ and $(\beta', \delta) \neq 0$. So without loss of generality, we assume $(\beta, \beta) = 0$. Now as $(\beta, \delta) \neq 0$ and $(\beta, k\delta) = (\beta, k\delta)_{\mathbb{F}} = k(\beta, \delta) \neq 0$, there are $r, s \in \{\pm 1\}$ such that $\beta + r\delta, \beta + sk\delta \in R$. Now we have

$$\begin{aligned} (\beta + sk\delta, \beta + sk\delta) &= (\beta + sk\delta, \beta + sk\delta)_{\mathbb{F}} = 2sk(\beta, \delta) \quad \text{and} \\ (\beta + r\delta, \beta + sk\delta) &= (\beta + r\delta, \beta + sk\delta)_{\mathbb{F}} = (sk + r)(\beta, \delta). \end{aligned}$$

If $k = \frac{-r}{s}$, then $k \in \{\pm 1\}$ and so we are done; otherwise, $\frac{sk+r}{sk} = \frac{2(\beta+r\delta, \beta+sk\delta)}{(\beta+sk\delta, \beta+sk\delta)}$ is an integer number. This implies that $k \in \{\pm 1\}$. \square

Definition 1.16. Suppose that $(A, (\cdot, \cdot), R)$ is a locally finite root supersystem. A subset Π of R is called an *integral base* for R if Π is a \mathbb{Z} -basis for A . An integral base Π of R is called a *base* for R if for each $\alpha \in R^\times$, there are $\alpha_1, \dots, \alpha_n \in \Pi$ (not necessarily distinct) and $r_1, \dots, r_n \in \{\pm 1\}$ such that $\alpha = r_1\alpha_1 + \dots + r_n\alpha_n$ and for all $1 \leq t \leq n$, $r_1\alpha_1 + \dots + r_t\alpha_t \in R^\times$.

Lemma 1.17. Suppose that $(A, (\cdot, \cdot), R)$ is an irreducible locally finite root supersystem of type X . Then R contains an integral base; in particular, A is a free abelian group. Moreover, if $X \neq A(\ell, \ell)$, R possesses a base.

Proof. Contemplating [9, Lem. 5.1] and [8, §10.2], we assume that $R_{ns} \neq \{0\}$ and take R to be one of the root supersystems listed in Theorems 1.9 or 1.10. In what

follows for index sets T and T' with $|T|, |T'| \geq 2$ and a positive integer ℓ , we use the following notations:

A_T	$\{\epsilon_i - \epsilon_j \mid i, j \in T\}$	BC_1	$\{0, \pm\epsilon_0, \pm 2\epsilon_0\}, \{0, \pm\delta_0, \pm 2\delta_0\}$
$A_{T'}$	$\{\delta_p - \delta_q \mid p, q \in T'\}$	B_T	$\{0, \pm\epsilon_i, \pm\epsilon_i \pm \epsilon_j \mid i, j \in T, i \neq j\}$
C_T	$\{\pm\epsilon_i \pm \epsilon_j \mid i, j \in T\}$	$B_{T'}$	$\{0, \pm\delta_p, \pm\delta_p \pm \delta_q \mid p, q \in T', p \neq q\}$
$C_{T'}$	$\{\pm\delta_p \pm \delta_q \mid p, q \in T'\}$	A_1	$\{0, \pm\epsilon_0\}, \{0, \pm\delta_0\}, \{0, \pm\gamma_0\}$
D_T	$B_T \cap C_T$	A_ℓ	$\{\delta_i - \delta_j \mid 1 \leq i, j \leq \ell + 1\}$
$BC_{T'}$	$B_{T'} \cup C_{T'}$	A_ℓ	$\{\epsilon_i - \epsilon_j \mid 1 \leq i, j \leq \ell + 1\}$
BC_T	$B_T \cup C_T$	G_2	$\{0, \pm(\epsilon_i - \epsilon_j), \pm(2\epsilon_i - \epsilon_j - \epsilon_t) \mid \{i, j, t\} = \{1, 2, 3\}\}$

In addition, we fix $t_0 \in T$ and $p_0 \in T'$ and consider the notations as in Theorems 1.9 and 1.10. We next take Π to be as in the following table:

type	Π
$\dot{A}(0, T)$	$\{\alpha^*, \epsilon_t - \epsilon_{t_0} \mid t \in T \setminus \{t_0\}\}$
$\dot{C}(0, T)$	$\{\alpha^*, 2\epsilon_{t_0}, \epsilon_t - \epsilon_{t_0} \mid t \in T \setminus \{t_0\}\}$
$\dot{A}(T, T')$	$\{\alpha^*, \epsilon_t - \epsilon_{t_0}, \delta_{t'} - \delta_{p_0} \mid t \in T \setminus \{t_0\}, t' \in T' \setminus \{p_0\}\}$
$A(\ell, \ell)$	$\{\epsilon_i - \epsilon_{i+1}, \omega_1^1 + \omega_2^1, \delta_r - \delta_{r+1} \mid 1 \leq i \leq \ell - 1, 1 \leq r \leq \ell\}$
$B(T, T')$	$\{\epsilon_{t_0}, \epsilon_t - \epsilon_{t_0}, \delta_p - \epsilon_{t_0} \mid t \in T \setminus \{t_0\}, p \in T'\}$
$BC(T, T')$	$\{\epsilon_{t_0}, \epsilon_t - \epsilon_{t_0}, \delta_p - \epsilon_{t_0} \mid t \in T \setminus \{t_0\}, p \in T'\}$
$BC(1, 1)$	$\{\epsilon_0, \epsilon_0 + \delta_0\}$
$BC(1, T)$	$\{\epsilon_0, \epsilon_{t_0}, \epsilon_t - \epsilon_{t_0} \mid t \in T \setminus \{t_0\}\}$
$D(T, T')$	$\{2\delta_{p_0}, \delta_p - \delta_{p_0}, \epsilon_t - \delta_{p_0} \mid p \in T' \setminus \{p_0\}, t \in T\}$
$C(T, T')$	$\{2\epsilon_{t_0}, \epsilon_t - \epsilon_{t_0}, \delta_p - \epsilon_{t_0} \mid t \in T \setminus \{t_0\}, p \in T'\}$
$B(1, T)$	$\{\epsilon_0, \epsilon_0 - \epsilon_t \mid t \in T\}$
$C(1, T')$	$\{\epsilon_0, \frac{1}{2}\epsilon_0 - \delta_p \mid p \in T'\}$
$AB(1, 3)$	$\{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \epsilon_3, \frac{1}{2}(\epsilon_0 - \epsilon_1 - \epsilon_2 - \epsilon_3)\}$
$D(1, T)$	$\{\epsilon_0, \frac{1}{2}\epsilon_0 - \epsilon_t \mid t \in T\}$
$B(T, 1)$	$\{\epsilon_0, \epsilon_0 - \epsilon_t \mid t \in T\}$
$G(1, 2)$	$\{\epsilon_0, \epsilon_0 - \epsilon_1 + \epsilon_2, 2\epsilon_1 - \epsilon_2 - \epsilon_3\}$
$D(2, 1, \lambda)$	$\{\epsilon_0, \delta_0, \frac{1}{2}\epsilon_0 + \frac{1}{2}\delta_0 + \frac{1}{2}\gamma_0\}$
$D(2, T)$	$\{\epsilon_0, \delta_0, \frac{1}{2}\epsilon_0 + \frac{1}{2}\delta_0 + \epsilon_{t_0}, \epsilon_t - \epsilon_{t_0} \mid t \in T \setminus \{t_0\}\}$

One can check that Π is an integral base for R and that if R is not of type $A(\ell, \ell)$, Π is a base for R . \square

Using the same argument as in Lemma 3.1 of [19] and contemplating Lemma 1.17, one has the following lemma:

Lemma 1.18. (i) Suppose that Π is a base for an irreducible locally finite root supersystem $(A, (\cdot, \cdot), R)$. Then for each finite subset $X \subseteq \Pi$, there is a finite subset $Y_X \subseteq \Pi$ such that $X \subseteq Y_X$ and the form restricted to $\langle Y_X \rangle$ is nondegenerate. Moreover, if X is connected, we can choose Y_X to be connected.

(ii) If Π is a connected integral base for a locally finite root supersystem R , then R is irreducible.

(iii) Suppose that R is an infinite irreducible locally finite root supersystem in an additive abelian group A . Then there is a base Π for R and a class $\{R_\gamma \mid \gamma \in \Gamma\}$ of finite irreducible \mathbb{Z} -linearly closed sub-supersystems of R of the same type as R such that R is the direct union of R_γ 's and for each $\gamma \in \Gamma$, $\Pi \cap R_\gamma$ is a base for R_γ . In particular, each finite subset of R lies in a finite \mathbb{Z} -linearly closed sub-supersystem.

2. STRUCTURE THEOREM

In this section, we give a description of the structure of extended affine root supersystems. The following proposition is a generalization of Proposition 5.9 of [5] to extended affine root supersystems.

Proposition 2.1. *Suppose that A is an additive abelian group equipped with a symmetric form. Consider the induced form (\cdot, \cdot) on $\bar{A} = A/A^0$ and suppose that $\bar{\cdot} : A \rightarrow \bar{A}$ is the canonical epimorphism. Assume that S is a subset of $A^\times := A \setminus A^0$ and set $B := \langle S \rangle$. If*

- $(\bar{B}, (\cdot, \cdot) |_{\bar{B} \times \bar{B}}, \bar{S} \cup \{0\})$ is a locally finite root supersystem,
- $S = -S$ and $\alpha - \frac{2(\alpha, \beta)}{(\beta, \beta)}\beta \in S$ for $\beta \in S_{re}^\times$ and $\alpha \in S$,
- for $\alpha \in S_{ns}$ and $\beta \in S$ with $(\alpha, \beta) \neq 0$, $\{\beta + \alpha, \beta - \alpha\} \cap S \neq \emptyset$,

then $R := S \cup ((S - S) \cap A^0)$ is a tame extended affine root supersystem in its \mathbb{Z} -span.

Proof. To show that R is a tame extended affine root supersystem, we just need to prove that the root string property holds. Suppose that $\alpha \in R_{re}^\times$ and $\beta \in R$.

Step 1. $\beta \notin R_{ns}^\times$: We know that $\bar{\alpha}, \bar{\beta}$ are two elements of the locally finite root supersystem $\bar{S} \cup \{0\}$. So using Lemma 1.18, there is a finite sub-supersystem Φ with $\bar{\alpha}, \bar{\beta} \in \Phi$ such that $(\mathbb{Z}\bar{\alpha} + \mathbb{Z}\bar{\beta}) \cap \bar{R} \subseteq \Phi$. Set

$$R_{\alpha, \beta} := R \cap (\mathbb{Z}\alpha + \mathbb{Z}\beta) \quad \text{and} \quad X_{\alpha, \beta} := R \cap (\beta + \mathbb{Z}\alpha).$$

Then as $(\bar{R}_{\alpha, \beta})_{re}$ is invariant under the reflections based on its nonzero elements, it is a subsystem of the finite root system Φ_{re} . Now we carry out the proof through the following two cases:

Case 1. $(\mathbb{Z}\alpha + \mathbb{Z}\beta) \cap A^0 \neq \{0\}$: In this case, $\bar{\alpha}, \bar{\beta}$ are \mathbb{Z} -linearly dependent elements of $(\bar{R}_{\alpha, \beta})_{re}$, so $(\bar{R}_{\alpha, \beta})_{re}$ is a finite root system of rank 1, in other words, it is either of type A_1 or BC_1 . Then there is $\delta \in A^0$ such that

$$X_{\alpha, \beta} \subseteq \begin{cases} \{0, \pm 1, \pm 2\}\alpha + \delta & \text{if } \bar{\beta} \in \{0, \pm 2\bar{\alpha}, \pm \bar{\alpha}\} \\ \{\beta, \beta \mp \alpha\} = \{\beta, r_\alpha(\beta)\} & \text{if } \bar{\alpha} \in \{\pm 2\bar{\beta}\}. \end{cases}$$

So the root string property holds if $\bar{\alpha} \in \{\pm 2\bar{\beta}\}$. If $\bar{\beta} \in \{0, \pm 2\bar{\alpha}, \pm \bar{\alpha}\}$, then $X_{\alpha, \beta} = Y\alpha + \delta$ where Y is a subset of $\{0, \pm 1, \pm 2\}$. As R is invariant under r_α , we have $Y = -Y$. If $2 \in Y$, then $\delta + \alpha = -r_{\delta+2\alpha}\alpha \in R$ that is $1 \in Y$. Also if $1 \in Y$, then $\delta + \alpha \in S$, and so $\delta = (\delta + \alpha) - \alpha \in (S - S) \cap A^0 \subseteq R$, that is $0 \in Y$. We conclude that either $Y = \{0, \pm 1, \pm 2\}$ or $Y = \{0, \pm 1\}$ and so the root string property holds.

Case 2. $(\mathbb{Z}\alpha + \mathbb{Z}\beta) \cap A^0 = \{0\}$: If $\bar{\alpha}, \bar{\beta}$ are \mathbb{Z} -linearly dependent, we get the result as in Case 1. We next suppose $\bar{\alpha}, \bar{\beta}$ are \mathbb{Z} -linearly independent. We claim that the form restricted to $\mathbb{Z}\alpha + \mathbb{Z}\beta$ is nondegenerate. We suppose that $r\alpha + s\beta$ is an element of the radical of the form on $\mathbb{Z}\alpha + \mathbb{Z}\beta$ and prove that $r = s = 0$. If either $r = 0$ or $s = 0$, we are done. So we assume $r, s \neq 0$ and get a contradiction. We have $r(\alpha, \alpha) + s(\beta, \alpha) = (r\alpha + s\beta, \alpha) = 0$ and $r(\alpha, \beta) + s(\beta, \beta) = (r\alpha + s\beta, \beta) = 0$. This implies that $(\bar{\alpha}, \bar{\beta})/(\bar{\alpha}, \bar{\alpha}) = -r/s$ and $(\bar{\alpha}, \bar{\beta})/(\bar{\beta}, \bar{\beta}) = -s/r$. But $\bar{\alpha}, \bar{\beta}$ are two \mathbb{Z} -linearly independent roots of the finite root system Φ_{re} , so we get $4 = (2r/s)(2s/r) = \frac{2(\bar{\alpha}, \bar{\beta})}{(\bar{\beta}, \bar{\beta})} \frac{2(\bar{\alpha}, \bar{\beta})}{(\bar{\alpha}, \bar{\alpha})} \in \{0, 1, 2, 3\}$, a contradiction. Therefore, the form restricted to $\mathbb{Z}\alpha + \mathbb{Z}\beta$ is nondegenerate. Also it is immediate that $R_{\alpha, \beta}$ satisfies (S1)-(S3) and (S5). We next take ϕ to be the restriction of “ $\bar{\cdot}$ ” to $(R_{\alpha, \beta})_{re}$. Since $(\mathbb{Z}\alpha + \mathbb{Z}\beta) \cap A^0 = \{0\}$, ϕ is an embedding into Φ_{re} ; in particular, $(R_{\alpha, \beta})_{re}$ is finite.

This in particular implies that the root string property holds in $R_{\alpha,\beta}$ (see [19, Lem. 3.10]) and so we are done in this case.

Step 2. $\beta \in R_{ns}^\times$: If $X_{\alpha,\beta} \cap (R \setminus R_{ns}^\times) \neq \emptyset$, then we get the result by Step 1. So we assume $X_{\alpha,\beta} \subseteq S_{ns}^\times$. Therefore, for $k \in \mathbb{Z}$, $k\alpha + \beta \in R$ implies that $(k\alpha + \beta, k\alpha + \beta) = 0$. Since $(\alpha, \alpha) \neq 0$, this gives $X_{\alpha,\beta} \subseteq \{\beta, r_\alpha\beta\}$. Since $r_\alpha\beta, \beta \in X_{\alpha,\beta}$, we get $X_{\alpha,\beta} = \{\beta, r_\alpha\beta\}$. If $(\beta, \alpha) = 0$, this gives $X_{\alpha,\beta} = \{\beta\}$, so the string property holds. If $(\beta, \alpha) \neq 0$, then $\beta + \alpha$ or $\beta - \alpha$ lies in $X_{\alpha,\beta}$, so $r_\alpha\beta$ is either $\beta + \alpha$ or $\beta - \alpha$; in both cases the root string property holds. This completes the proof. \square

In [12, §3] and [3, Thm. 1.13], the authors give the structure of an affine reflection system i.e., an extended affine root supersystem whose set of nonsingular roots is $\{0\}$. In the following theorem, we give the structure of extended affine root supersystems. We see that the notion of extended affine root systems is in fact a generalized notion of root systems extended by an abelian group introduced by Y. Yoshii [17]. More precisely, we show that associate to each extended affine root supersystem $(A, (\cdot, \cdot), R)$ of type X , there is a locally finite root supersystem \dot{R} as well as a class $\{S_\alpha\}_{\alpha \in \dot{R}}$ of subsets of A^0 such that $R = \cup_{\alpha \in \dot{R}} (\alpha + S_\alpha)$. If $X \neq A(\ell, \ell), C(1, 2), C(T, 2), BC(1, 1)$, then the interaction of S_α 's results in a nice characterization of R .

In what follows by a *reflectable set* for a locally finite root system S , we mean a subset Π of $S \setminus \{0\}$ such that $W_\Pi(\Pi)$ coincides with the set of nonzero reduced roots $S_{red}^\times = S \setminus \{2\alpha \mid \alpha \in S\}$, in which W_Π is the subgroup of the Weyl group generated by r_α for all $\alpha \in \Pi$; see [3]. We also recall from [7] that a *symmetric reflection subspace* (or s.r.s for short) of an additive abelian group A is a nonempty subset X of A satisfying $X - 2X \subseteq X$; we mention that a symmetric reflection subspace satisfies $X = -X$. A symmetric reflection subspace X of an additive abelian group A is called a *pointed reflection subspace* (or p.r.s for short) if $0 \in X$. Before stating the structure theorem of extended affine root supersystems, we make a convention that if \dot{R} is a locally finite root supersystem with decomposition $\dot{R}_{re} = \oplus_{i=1}^n \dot{R}_{re}^i$ of \dot{R}_{re} into irreducible subsystems, by $\dot{R}_*, * = sh, lg, ex$, we mean $\cup_{i=1}^n (\dot{R}_{re}^i)_*$.

Theorem 2.2. *Suppose that $(\dot{A}, (\cdot, \cdot), \dot{R})$ is an irreducible locally finite root supersystem of type X with $\dot{R}_{ns} \neq \{0\}$, as in Theorems 1.9 and 1.10 and A^0 is an additive abelian group. Extend the form (\cdot, \cdot) to the form (\cdot, \cdot) on $\dot{A} \oplus A^0$ whose radical is A^0 .*

(i) *Suppose that $X \neq A(\ell, \ell), BC(T, T'), C(T, T'), C(1, T)$, F is a subgroup of A^0 and S is a pointed reflection subspace of A^0 such that*

$$\begin{aligned} \langle S \rangle &= A^0, \quad F + S \subseteq S, \quad 2S + F \subseteq F \quad \text{and} \\ S &= F \text{ if } X \neq B(T, T'), B(T, 1), B(1, T). \end{aligned}$$

Then

$$R := (S - S) \cup (\dot{R}_{sh} + S) \cup ((\dot{R}^\times \setminus \dot{R}_{sh}) + F)$$

is a tame irreducible extended affine root supersystem of type X . Conversely, each tame irreducible extended affine root supersystem of type X arises in this manner.

(ii) *Suppose that $X = BC(1, T), BC(T, T'), |T|, |T'| > 1$, F is a subgroup of A^0 , S is a pointed reflection subspace of A^0 and E_1, E_2 are two symmetric reflection*

subspaces of A^0 such that

$$\langle S \rangle = A^0, \{\sigma + \tau, \sigma - \tau\} \cap (E_1 \cup E_2) \neq \emptyset; \quad \sigma, \tau \in F,$$

$$\begin{aligned} F + S &\subseteq S, \quad 2S + F \subseteq F, \quad 2F + E_i \subseteq E_i \text{ (if } (\dot{R}_{re}^i)_{lg} \neq \emptyset), \quad F + E_i \subseteq F, \\ S + E_i &\subseteq S, \quad E_i + 4S \subseteq E_i \text{ (} i = 1, 2). \end{aligned}$$

Then

$$R := (S - S) \cup (\dot{R}_{sh} + S) \cup (\dot{R}_{ex}^1 + E_1) \cup (\dot{R}_{ex}^2 + E_2) \cup ((\dot{R}_{lg} \cup \dot{R}_{ns}^\times) + F)$$

is a tame irreducible extended affine root supersystem of type X ; conversely each tame irreducible extended affine root supersystem of type X arises in this manner.

(iii) Suppose that $X = C(1, T')$, $|T'| > 2$, F is a subgroup of A^0 , S is a pointed reflection subspace of A^0 and L a symmetric reflection subspace of A^0 such that

$$\langle S \rangle = A^0, \{\sigma + \tau, \sigma - \tau\} \cap (S \cup L) \neq \emptyset; \quad \sigma, \tau \in F,$$

$$F + S \subseteq F, \quad L + F \subseteq F, \quad L + 2F \subseteq L.$$

Then

$$R = (S - S) \cup (\dot{R}_{sh}^1 + S) \cup ((\dot{R}_{sh}^2 \cup \dot{R}_{ns}^\times) + F) \cup (\dot{R}_{lg}^2 + L)$$

is a tame irreducible extended affine root supersystem of type X . Conversely, each tame irreducible extended affine root supersystem of type $C(1, T')$, $|T'| > 2$, arises in this manner.

(iv) Suppose that $X = C(T, T')$, $|T| \geq 2, |T'| > 2$, F is a subgroup of A^0 , L_1 is a pointed reflection subspace of A^0 and L_2 is a symmetric reflection subspace of A^0 such that

$$\langle S \rangle = A^0, \{\sigma + \tau, \sigma - \tau\} \cap (L_1 \cup L_2) \neq \emptyset; \quad \sigma, \tau \in F,$$

$$L_i + F \subseteq F, \quad L_i + 2F \subseteq L_i \quad (i = 1, 2).$$

Then

$$R = F \cup ((\dot{R}_{sh} \cup \dot{R}_{ns}^\times) + F) \cup ((\dot{R}_{re}^1)_{lg} + L_1) \cup ((\dot{R}_{re}^2)_{lg} + L_2)$$

is a tame irreducible extended affine root supersystem of type X . Conversely, each tame irreducible extended affine root supersystem of type $C(T, T')$, $|T| \geq 2, |T'| > 2$, arises in this manner.

Proof. Suppose that $(A, (\cdot, \cdot), R)$ is a tame irreducible extended affine root supersystem of type X with $R_{ns} \neq \{0\}$, then by Proposition 1.11(iii), $(\bar{A}, (\cdot, \cdot), \bar{R})$ is a locally finite root supersystem. Fix a subset $\bar{\Pi}$ of \bar{R} such that $\bar{\Pi}$ is the corresponding \mathbb{Z} -basis for \bar{A} as introduced in Lemma 1.17. Take $\dot{A} := \langle \bar{\Pi} \rangle$ as well as $\dot{R} := \{\dot{a} \in \dot{A} \mid \exists \eta \in A^0; \dot{a} + \eta \in R\}$. One can see that $A = \dot{A} \oplus A^0$, and that \dot{R} is a locally finite root supersystem in \dot{A} isomorphic to \bar{R} . Without loss of generality, by multiplying the form (\cdot, \cdot) to a nonzero scalar, we may assume \dot{R} is one of the locally finite root supersystems as in Theorems 1.9 and 1.10 with the decomposition $\dot{R}_{re} = \bigoplus_{i=1}^n \dot{R}_{re}^i$ for \dot{R}_{re} into irreducible subsystems and that $\bar{\Pi}$ is as in Lemma 1.17.

Claim 1. If \dot{R} is of type $X \neq A(\ell, \ell), C(T, T'), C(1, T)$, then R contains a reflectable set for \dot{R}_{re} . We note that if \dot{R} is of imaginary type, then $\bar{\Pi} \cap \dot{R}_{re} \subset R$ is a reflectable set for \dot{R}_{re} . So we suppose that \dot{R} is of real type and carry out the proof through the following cases:

- Case 1. \dot{R} is of type $AB(1, 3)$: In this case, since $\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \epsilon_3 \in \dot{R} \cap R$, we get that $\epsilon_1 + \epsilon_2 = r_{\epsilon_2 - \epsilon_3} r_{\epsilon_2 - \epsilon_1} r_{\epsilon_3}(\epsilon_2 - \epsilon_3) \in \dot{R} \cap R$. We note that for $\dot{\alpha} := \epsilon_1 + \epsilon_2, \dot{\beta} := \epsilon_3, \dot{\gamma} := \frac{1}{2}(\epsilon_0 - \epsilon_1 - \epsilon_2 - \epsilon_3) \in R \cap \dot{R}$, we have $\dot{\alpha} - \dot{\gamma}, \dot{\beta} - \dot{\gamma} \notin \dot{R}$. Therefore $\bar{\alpha} - \bar{\gamma}, \bar{\beta} - \bar{\gamma} \notin \bar{R}$ and so $\dot{\alpha} - \dot{\gamma}, \dot{\beta} - \dot{\gamma} \notin R$. This together with the fact that $(\dot{\alpha}, \dot{\gamma}) \neq 0$ and $(\dot{\beta}, \dot{\gamma}) \neq 0$, implies that $\dot{\alpha} + \dot{\gamma}, \dot{\beta} + \dot{\gamma} \in R$, and so $\dot{\eta} := \dot{\alpha} + \dot{\gamma}, \dot{\zeta} := \dot{\beta} + \dot{\gamma} \in \dot{A} \cap R \subseteq R \cap \dot{R}$. Again as $(\dot{\eta}, \dot{\zeta}) \neq 0$, the same argument as above implies that $\epsilon_0 = \dot{\eta} + \dot{\zeta} \in R \cap \dot{R}$. So we are done as $\{\epsilon_0, \epsilon_3, \epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3\}$ is a reflectable set for \dot{R}_{re} .
- Case 2. \dot{R} is of type $D(2, 1, \lambda)$: We know that $\dot{\eta} := \frac{1}{2}\epsilon_0 + \frac{1}{2}\delta_0 + \frac{1}{2}\gamma_0, \epsilon_0, \delta_0 \in R \cap \dot{R}$. Since $\dot{\eta} + \epsilon_0, \dot{\eta} + \delta_0 \notin \dot{R}$, we get that $\bar{\eta} + \bar{\epsilon}_0, \bar{\eta} + \bar{\delta}_0 \notin \bar{R}$ and so $\dot{\eta} + \epsilon_0, \dot{\eta} + \delta_0 \notin R$. Also as $(\dot{\eta}, \epsilon_0) \neq 0$ and $(\dot{\eta}, \delta_0) \neq 0$, we have $\dot{\zeta} := \dot{\eta} - \epsilon_0 = -\frac{1}{2}\epsilon_0 + \frac{1}{2}\delta_0 + \frac{1}{2}\gamma_0, \dot{\xi} := \dot{\eta} - \delta_0 = \frac{1}{2}\epsilon_0 - \frac{1}{2}\delta_0 + \frac{1}{2}\gamma_0 \in R \cap \dot{A}$. Again as $(\dot{\zeta}, \dot{\xi}) \neq 0$ and $\dot{\xi} - \dot{\zeta} \notin \dot{R}$, we get that $\gamma_0 = \dot{\xi} + \dot{\zeta} \in \dot{A} \cap R$ and so $\gamma_0 \in \dot{R} \cap R$. Therefore, $\{\epsilon_0, \delta_0, \gamma_0\}$, which is a reflectable set for \dot{R}_{re} , is contained in R .
- Case 3. \dot{R} is of type $D(2, T)$: Using the same argument as in the previous case, we get that $2\epsilon_{t_0} \in \dot{R} \cap R$ and so $\{\epsilon_0, \delta_0, 2\epsilon_{t_0}, \epsilon_t - \epsilon_{t_0} \mid t \in T \setminus \{t_0\}\}$, which is a reflectable set for \dot{R}_{re} , is contained in R .
- Case 4. \dot{R} is of type $D(T, T')$: Since for $i \in T$, $\epsilon_i - \delta_{p_0}, 2\delta_{p_0} \in R \cap \dot{R}$ and $\epsilon_i - \delta_{p_0} - 2\delta_{p_0} \notin \dot{R}$, as above one concludes that $\epsilon_i + \delta_{p_0} \in R \cap \dot{R}$. Now for $i, j \in T$ with $i \neq j$, we have $\dot{\alpha} := \epsilon_i - \delta_{p_0}, \dot{\beta} := \epsilon_i + \delta_{p_0}, \dot{\gamma} := \epsilon_j - \delta_{p_0} \in R \cap \dot{R}$ with $(\dot{\alpha}, \dot{\gamma}) \neq 0$ and $(\dot{\beta}, \dot{\gamma}) \neq 0$, but $\dot{\alpha} + \dot{\gamma}, \dot{\beta} - \dot{\gamma} \notin \dot{R}$, so as above, we get that $\epsilon_i + \epsilon_j, \epsilon_i - \epsilon_j \in R \cap \dot{R}$. This completes the proof in this case as $\{2\delta_{p_0}, \delta_p - \delta_{p_0}, \epsilon_i \pm \epsilon_j \mid p \in T' \setminus \{p_0\}, i \neq j \in T\} \subseteq R$ is a reflectable set for \dot{R}_{re} .
- Case 5. \dot{R} is of type $D(1, T)$: We know that for $t \in T$, $\epsilon_0, \frac{1}{2}\epsilon_0 - \epsilon_t \in \dot{R} \cap R$ and that $\epsilon_0 + (\frac{1}{2}\epsilon_0 - \epsilon_t) \notin \dot{R}$, so as before, we get that $\frac{1}{2}\epsilon_0 + \epsilon_t \in R \cap \dot{R}$. Using the same argument as in the previous case, we get that $\epsilon_r \pm \epsilon_s \in R \cap \dot{R}$ for all $r, s \in T$ with $r \neq s$. This completes the proof in this case.
- Case 6. \dot{R} is of type $B(T, T'), BC(T, T'), B(1, T), B(T, 1), G(1, 2)$: In these cases, for $\dot{\Pi}_{re} := \dot{\Pi} \cap \dot{R}_{re}$ and $\dot{\Pi}_{ns} := \dot{\Pi} \cap \dot{R}_{ns}$, the set $\dot{\Pi}_{re} \cup ((\dot{\Pi}_{ns} \pm \dot{\Pi}) \cap (\dot{R}_{re})_{red}^\times)$, which is (as above) a subset of R , is a reflectable set for \dot{R}_{re} .

Claim 2. If $X = C(T, T'), C(1, T')$, then $\dot{R}_{re}^\times \setminus (\dot{R}_{re}^2)_{lg} \subseteq R$: We know that $\dot{\Pi} \subseteq R \cap \dot{R}$. So as in Case 5 of the the proof of Claim 1, we get that $\pm \delta_p \pm \delta_q \in R_{re}$ for all $p, q \in T'$ with $p \neq q$. Moreover, if $X = C(T, T')$, then for $t \in T \setminus \{t_0\}$, since $2\epsilon_{t_0}, \epsilon_t - \epsilon_{t_0} \in R \cap \dot{R}$, we have that

$$\begin{aligned} \epsilon_t + \epsilon_{t_0} &= r_{2\epsilon_{t_0}}(\epsilon_t - \epsilon_{t_0}) \in \dot{R} \cap R, & \epsilon_r - \epsilon_t &= r_{\epsilon_t - \epsilon_{t_0}}(\epsilon_r - \epsilon_{t_0}) \in \dot{R} \cap R, \\ \epsilon_r + \epsilon_t &= r_{\epsilon_t - \epsilon_{t_0}}(\epsilon_r + \epsilon_{t_0}) \in \dot{R} \cap R, & 2\epsilon_t &= r_{\epsilon_{t_0} - \epsilon_t}(2\epsilon_{t_0}) \in \dot{R} \cap R, \end{aligned}$$

for all $r, t \in T$ with $r, t \in T \setminus \{t_0\}$ and $r \neq t$. This completes the proof of the claim.

Claim 3. For $\dot{\alpha} \in \dot{R}$, set

$$S_{\dot{\alpha}} := \{\eta \in A^0 \mid \dot{\alpha} + \eta \in R\}.$$

Then we have

$$(2.4) \quad \begin{cases} 0 \in S_{\dot{\alpha}} & \text{if } X \neq A(\ell, \ell), C(T, T'), C(1, T') \text{ \& } \dot{\alpha} \in (\dot{R}_{re})_{red}, \\ 0 \in S_{\dot{\alpha}} & \text{if } X = C(T, T'), C(1, T') \text{ \& } \dot{\alpha} \in \dot{R}_{re}^\times \setminus (\dot{R}_{re}^2)_{lg}, \end{cases}$$

and that

$$(2.5) \quad \begin{aligned} & \text{if } X \neq A(\ell, \ell), C(1, 2), C(T, 2), S_{\dot{\alpha}} = S_{\dot{\beta}} \text{ for all} \\ & \dot{\alpha}, \dot{\beta} \in \dot{R}_{re}^i \setminus \{0\} \ (1 \leq i \leq n) \text{ with } (\dot{\alpha}, \dot{\alpha}) = (\dot{\beta}, \dot{\beta}) : \end{aligned}$$

To show this, suppose that $\dot{\alpha} \in \dot{R}_{re}^\times, \dot{\beta} \in \dot{R}, \eta \in S_{\dot{\alpha}}$ and $\zeta \in S_{\dot{\beta}}$, then

$$r_{\dot{\alpha}+\eta}(\dot{\beta} + \zeta) = \dot{\beta} + \zeta - \frac{2(\dot{\alpha}, \dot{\beta})}{(\dot{\alpha}, \dot{\alpha})}(\dot{\alpha} + \eta) = r_{\dot{\alpha}}(\dot{\beta}) + (\zeta - \frac{2(\dot{\alpha}, \dot{\beta})}{(\dot{\alpha}, \dot{\alpha})}\eta).$$

This means that

$$(2.6) \quad S_{\dot{\beta}} - \frac{2(\dot{\alpha}, \dot{\beta})}{(\dot{\alpha}, \dot{\alpha})}S_{\dot{\alpha}} \subseteq S_{r_{\dot{\alpha}}(\dot{\beta})}; \quad (\dot{\alpha} \in \dot{R}_{re}^\times, \dot{\beta} \in \dot{R}).$$

This in turn implies that

$$(2.7) \quad S_{\dot{\beta}} - 2S_{\dot{\beta}} \subseteq S_{-\dot{\beta}}; \quad (\dot{\beta} \in \dot{R}_{re}^\times).$$

Now suppose that \dot{R} is of type $X = C(T, T'), C(1, T')$. Using Claim 2, we have $0 \in S_{\dot{\alpha}}$ for $\dot{\alpha} \in \dot{R}_{re}^\times \setminus (\dot{R}_{re}^2)_{lg}$. We next mention that for a locally finite root system of type C with rank greater than 2, roots of the same length are conjugate under the subgroup of the Weyl group generated by the reflection based on the short roots, therefore in this case, by (2.6), we get $S_{\dot{\alpha}} = S_{\dot{\beta}}$ for all $\dot{\alpha}, \dot{\beta} \in \dot{R}_{re}^i \setminus \{0\}$ ($1 \leq i \leq n$) with $(\dot{\alpha}, \dot{\alpha}) = (\dot{\beta}, \dot{\beta})$.

Next suppose that $X \neq A(\ell, \ell), C(T, T'), C(1, T')$. Using Claim 1, we get that R contains a reflectable set for \dot{R}_{re} , say \dot{B} . Now for $\dot{\alpha} \in (\dot{R}_{re})_{red} \setminus \{0\}$, there are $\dot{\alpha}_1, \dots, \dot{\alpha}_{t+1} \in \dot{B} \subseteq R \cap \dot{R}$ such that $r_{\dot{\alpha}_1} \cdots r_{\dot{\alpha}_t}(\dot{\alpha}_{t+1}) = \dot{\alpha}$, so as R and \dot{R} are closed under the reflection actions, we get that $\dot{\alpha} \in R \cap \dot{R}$; in particular $0 \in S_{\dot{\alpha}}$ for $\dot{\alpha} \in (\dot{R}_{re})_{red}$. These all together with (2.6) and the fact that for a locally finite root system, the roots of the same length are conjugate under the Weyl group action complete the proof.

Claim 4. Suppose that $X \neq A(\ell, \ell), C(1, 2), C(T, 2), BC(1, 1)$. Fix a nonzero $\dot{\delta}^* \in \dot{R}_{ns} \cap \dot{\Pi} \subseteq \dot{R} \cap R$. Consider (2.5) and set

$$F := S_{\dot{\delta}^*} \quad \text{and} \quad \begin{cases} S_i := S_{\dot{\alpha}} & \dot{\alpha} \in (\dot{R}_{re}^i)_{sh} \\ L_i := S_{\dot{\alpha}} & \dot{\alpha} \in (\dot{R}_{re}^i)_{lg} \\ E_i := S_{\dot{\alpha}} & \dot{\alpha} \in (\dot{R}_{re}^i)_{ex} \end{cases}$$

for $1 \leq i \leq n$. Then

$$(2.8) \quad \begin{aligned} & S_i \text{ is a p.r.s. of } A^0 \\ & E_i \text{ is a s.r.s. of } A^0 \quad \text{if } (\dot{R}_{re}^i)_{ex} \neq \emptyset, \\ & L_i \text{ is a p.r.s. of } A^0 \quad \text{if } X \neq C(1, T'), C(T, T') \text{ and } (\dot{R}_{re}^i)_{lg} \neq \emptyset, \\ & L_2 \text{ is a s.r.s. of } A^0 \quad \text{if } X = C(1, T'), C(T, T'); |T'| > 2, \\ & L_1 \text{ is a p.r.s. of } A^0 \quad \text{if } X = C(T, T'); |T'| > 2 \end{aligned}$$

and

$$(2.9) \quad \begin{aligned} (a) \quad & S_i + L_i \subseteq S_i, \ L_i + \rho_i S_i \subseteq L_i \quad \text{if } (\dot{R}_{re}^i)_{lg} \neq \emptyset, \\ (b) \quad & S_i + E_i \subseteq S_i, \ E_i + 4S_i \subseteq E_i \quad \text{if } \dot{R}_{re}^i = BC_1, \\ (c) \quad & L_i + E_i \subseteq L_i, \ E_i + 2L_i \subseteq E_i \quad \text{if } \dot{R}_{re}^i = BC_P \ (|P| \geq 2), \end{aligned}$$

in which

$$\rho_i := (\dot{\beta}, \dot{\beta})/(\dot{\alpha}, \dot{\alpha}), \quad (\dot{\alpha} \in (\dot{R}_{re}^i)_{sh}, \dot{\beta} \in (\dot{R}_{re}^i)_{lg} \text{ if } (\dot{R}_{re}^i)_{lg} \neq \emptyset :$$

We immediately get (2.8) using (2.4), (2.5) and (2.7). Now if $(\dot{R}_{re}^i)_{lg} \neq \emptyset$, there are $\dot{\alpha} \in (\dot{R}_{re}^i)_{lg}$ and $\dot{\beta} \in (\dot{R}_{re}^i)_{sh}$ such that $2(\dot{\alpha}, \dot{\beta})/(\dot{\beta}, \dot{\beta}) = \rho_i$ and $2(\dot{\beta}, \dot{\alpha})/(\dot{\alpha}, \dot{\alpha}) = 1$, so using (2.6) and (2.5), we get (2.9)(a). If \dot{R}_{re}^i is of type BC_P for some index set P with $|P| \geq 2$, one finds $\dot{\alpha} \in (\dot{R}_{re}^i)_{ex}$ and $\dot{\beta} \in (\dot{R}_{re}^i)_{lg}$ such that $2(\dot{\alpha}, \dot{\beta})/(\dot{\beta}, \dot{\beta}) = 2$ and $2(\dot{\beta}, \dot{\alpha})/(\dot{\alpha}, \dot{\alpha}) = 1$ and so we get (2.9)(c). We similarly have (2.9)(b) as well.

Claim 5. F is a subgroup of A^0 ,

$$(2.10) \quad F + 2S_i \subseteq F; \quad (1 \leq i \leq n).$$

and for each $\dot{\delta} \in \dot{R}_{ns}^\times$, we have $S_{\dot{\delta}} = F$. Also

$$(2.11) \quad F = \begin{cases} S_i & \text{if } X \neq C(1, T'), BC(T, T'), B(T, T'), B(T, 1), B(1, T); \quad 1 \leq i \leq n, \\ L_i & \text{if } X \neq C(T, T'), C(1, T) \text{ \& } (\dot{R}_{re}^i)_{lg} \neq \emptyset; \quad 1 \leq i \leq n, \\ E_i & \text{if } X \neq BC(T, T') \text{ \& } (\dot{R}_{re}^i)_{ex} \neq \emptyset; \quad 1 \leq i \leq n, \\ S_2 & \text{if } X = C(1, T') : \end{cases}$$

We know that \dot{W} , the Weyl group of \dot{R} , is generated by the reflections based on nonzero reduced roots and that if $X = C(T, T'), C(1, T'), |T'| > 2$, nonsingular roots are conjugate with $\dot{\delta}^*$ under the subgroup of \dot{W} generated by the reflections based on the elements of $(\dot{R}_{re}^1)_{sh} \cup (\dot{R}_{re}^2)_{sh}$. So (2.6) together with (2.4) and the fact that $\alpha \in R$ if and only if $-\alpha \in R$, implies that

$$(2.12) \quad \begin{cases} S_{\pm \dot{w}\dot{\delta}^*} = S_{\pm \dot{\delta}^*} = \pm F & \dot{w} \in \dot{W} \\ 0 \in S_{\dot{\alpha}} & \dot{\alpha} \in \dot{R}_{ns}. \end{cases}$$

Also one can easily see that

$$(2.13) \quad (\dot{W}\dot{\delta}^* - \dot{W}\dot{\delta}^*) \cap \dot{R}^\times = \begin{cases} \dot{R}_{re} \setminus ((\dot{R}_1)_{sh} \cup (\dot{R}_2)_{sh}) & X = B(T, T'), BC(T, T'), B(1, T), B(T, 1), \\ (\dot{R}_{re}^1)_{ex} \cup \dot{R}_{re}^2 \setminus \{0\} & X = G(1, 2), \\ \dot{R}_{re}^\times & \text{otherwise.} \end{cases}$$

Moreover, if $1 \leq i \leq n$, we have

$$(2.14) \quad (\dot{W}\dot{\delta}^* + (\dot{R}_{re}^i)_{sh}) \cap \dot{R}^\times = \begin{cases} (\dot{R}_{re}^j)_{sh} & X = B(T, T'), BC(T, T'), B(1, T), B(T, 1), \\ (\dot{R}_{re}^2)_{sh} & X = G(1, 2), i = 1, \\ \dot{W}\dot{\delta}^* \cup (\dot{R}_{re}^1)_{sh} & X = G(1, 2), i = 2, \\ \dot{W}\dot{\delta}^* & \text{otherwise} \end{cases}$$

and

$$(2.15) \quad \begin{aligned} &(\dot{W}\dot{\delta}^* + (\dot{R}_{re}^i)_{lg}) \cap \dot{R}^\times = \dot{W}\dot{\delta}^*; \text{ if } (\dot{R}_{re}^i)_{lg} \neq \emptyset, \\ &(\dot{W}\dot{\delta}^* + (\dot{R}_{re}^i)_{ex}) \cap \dot{R}^\times = \dot{W}\dot{\delta}^*; \text{ if } (\dot{R}_{re}^i)_{ex} \neq \emptyset. \end{aligned}$$

We next note that

$$\text{if } X \neq A(\ell, \ell), BC(T, T'), C(T, T'), C(1, T) \text{ and } \dot{\alpha} \in \dot{R}_{ns}, \dot{\beta} \in \dot{R} \text{ with } (\dot{\alpha}, \dot{\beta}) \neq 0, \text{ then there is a unique } r_{\dot{\alpha}, \dot{\beta}} \in \{\pm 1\} \text{ with } \dot{\alpha} + r_{\dot{\alpha}, \dot{\beta}} \dot{\beta} \in \dot{R}$$

and that

if $X = BC(T, T'), C(T, T'), C(1, T)$ and $\dot{\alpha} \in \dot{R}_{ns}, \dot{\beta} \in \dot{R}_{re}$ with $(\dot{\alpha}, \dot{\beta}) \neq 0$, then there is a unique $s_{\dot{\alpha}, \dot{\beta}} \in \{\pm 1\}$ with $\dot{\alpha} + s_{\dot{\alpha}, \dot{\beta}} \dot{\beta} \in \dot{R}$. Moreover,

$$(2.16) \quad \text{for } \dot{\beta}, \dot{\gamma} \in \dot{R}_{ns} \text{ with } \dot{\beta} + \dot{\gamma}, \dot{\beta} - \dot{\gamma} \in \dot{R}, \text{ we have}$$

$$\dot{\beta} + \dot{\gamma}, \dot{\beta} - \dot{\gamma} \in \begin{cases} \dot{R}_{ex} & \text{if } X = BC(T, T'), \\ \dot{R}_{lg} & \text{if } X = C(T, T'), \\ (\dot{R}_{re}^1)_{sh} \cup (\dot{R}_{re}^2)_{lg} & \text{if } X = C(1, T); \end{cases}$$

(see Lemmas 1.3 and 1.15). Therefore,

$$S_{\dot{\alpha}} + r_{\dot{\alpha}, \dot{\beta}} S_{\dot{\beta}} \subseteq S_{\dot{\alpha} + r_{\dot{\alpha}, \dot{\beta}} \dot{\beta}} \quad (\dot{\alpha} \in \dot{R}_{ns}, \dot{\beta} \in \dot{R}, (\dot{\alpha}, \dot{\beta}) \neq 0)$$

$$X \neq BC(T, T'), C(T, T'), C(1, T)$$

and

$$(2.17) \quad S_{\dot{\alpha}} + s_{\dot{\alpha}, \dot{\beta}} S_{\dot{\beta}} \subseteq S_{\dot{\alpha} + s_{\dot{\alpha}, \dot{\beta}} \dot{\beta}} \quad (\dot{\alpha} \in \dot{R}_{ns}, \dot{\beta} \in \dot{R}_{re}, (\dot{\alpha}, \dot{\beta}) \neq 0)$$

$$X = BC(T, T'), C(T, T'), C(1, T).$$

Now we drew the attention of the readers to the point that if $\dot{\alpha}, \dot{\beta} \in \dot{R}_{ns}^\times$ with $\dot{\alpha} + \dot{\beta}, \dot{\alpha} - \dot{\beta} \in \dot{R}$, although for $\sigma \in S_{\dot{\alpha}}, \tau \in S_{\dot{\beta}}$, there is $r \in \{\pm 1\}$ with $(\dot{\alpha} + \sigma) + r(\dot{\beta} + \tau) \in \dot{R}$, we cannot conclude that both $(\dot{\alpha} + \sigma) + (\dot{\beta} + \tau)$ and $(\dot{\alpha} + \sigma) - (\dot{\beta} + \tau)$ are elements of \dot{R} . Considering this together with (2.16) and using (2.13), we have for $X \neq A(\ell, \ell), C(1, 2), C(T, 2)$ that

$$(2.18) \quad F - F \subseteq \begin{cases} E_i & \text{if } X = B(1, T), B(T, T'), B(T, 1), (\dot{R}_{re}^i)_{ex} \neq \emptyset, \\ L_i & \text{if } X = B(1, T), B(T, T'), B(T, 1), (\dot{R}_{re}^i)_{lg} \neq \emptyset, \\ L_i & \text{if } X = BC(T, T') \text{ and } (\dot{R}_{re}^i)_{lg} \neq \emptyset, \\ S_i & \text{if } X = C(T, T'), |T'| > 2, \\ S_2 & \text{if } X = C(1, T'), |T'| > 2, \\ E_1, S_2, L_2 & \text{if } X = G(1, 2), \\ S_i & \text{if } X = \text{Remain types under consideration}, \\ L_i & \text{if } X = \text{Remain types under consideration}, (\dot{R}_{re}^i)_{lg} \neq \emptyset, \\ E_i & \text{if } X = \text{Remain types under consideration}, (\dot{R}_{re}^i)_{ex} \neq \emptyset \end{cases}$$

for $1 \leq i \leq n$. Also by (2.14), we have

$$(2.19) \quad F + S_i \subseteq \begin{cases} S_j & \text{if } X = B(T, T'), BC(T, T'), B(1, T), B(T, 1), G(1, 2), \{i, j\} = \{1, 2\}, \\ F & \text{if } X \neq B(T, T'), BC(T, T'), B(1, T), B(T, 1), G(1, 2), 1 \leq i \leq n, \\ F & \text{if } X = G(1, 2), i = 2. \end{cases}$$

In addition, by (2.15) and (2.17), we have

$$(2.20) \quad F + L_i \subseteq F \text{ (if } (\dot{R}_{re}^i)_{lg} \neq \emptyset) \quad \text{and} \quad F + E_i \subseteq F \text{ (if } (\dot{R}_{re}^i)_{ex} \neq \emptyset).$$

In particular, since $0 \in F$, (2.18) imply that

$$(2.21) \quad F = \begin{cases} L_i & \text{if } (\dot{R}_{re}^i)_{lg} \neq \emptyset \text{ and } X \neq C(T, T'), C(1, T) \\ E_i & \text{if } (\dot{R}_{re}^i)_{ex} \neq \emptyset, X \neq BC(T, T'). \end{cases}$$

Moreover, (2.19) implies that

$$(2.22) \quad \text{if } X = B(T, T'), BC(T, T'), B(1, T), B(T, 1), G(1, 2), \text{ then } S_1 = S_2,$$

so for $1 \leq i \leq n$, using (2.19), we have

$$(2.23) \quad F + S_i \subseteq \begin{cases} S_i & \text{if } X = B(T, T'), BC(T, T'), B(1, T), B(T, 1), G(1, 2), \\ F & \text{if } X \neq B(T, T'), BC(T, T'), B(1, T), B(T, 1). \end{cases}$$

We also have using (2.8) and (2.9) that $L_i \subseteq S_i$ if $(\dot{R}_{re}^i)_{lg} \neq \emptyset$, $E_i \subseteq L_i$ if $(\dot{R}_{re}^i)_{ex} \neq \emptyset$ and $E_i \subseteq S_i$ if \dot{R}_{re}^i is of type BC_1 . So by (2.18) and (2.22), we have

$$(2.24) \quad F - F \subseteq \begin{cases} S_i & \text{if } X \neq C(1, T), BC(1, 1); 1 \leq i \leq n, \\ L_i & \text{if } X \neq C(T, T'), C(1, T) \text{ and } (\dot{R}_{re}^i)_{lg} \neq \emptyset; 1 \leq i \leq n, \\ E_i & \text{if } X \neq BC(T, T') \text{ and } (\dot{R}_{re}^i)_{ex} \neq \emptyset; 1 \leq i \leq n, \\ S_2 & \text{if } X = C(1, T'), |T'| > 2. \end{cases}$$

Therefore, we have

$$\left\{ \begin{array}{l} F - F \stackrel{(2.24)}{\subseteq} S_i \stackrel{(2.23)}{\subseteq} F \quad \begin{array}{l} X \neq BC(T, T'), B(T, T'), B(1, T), \\ B(T, 1), C(1, T'), \quad 1 \leq i \leq n, \end{array} \\ \\ F - F \stackrel{(2.24)}{\subseteq} L_2 \stackrel{(2.21)}{\subseteq} F \quad \begin{array}{l} X = BC(1, T), BC(T, T'), B(T, T'), B(1, T) \\ B(T, 1); |T|, |T'| \geq 2, \end{array} \\ \\ F - F \stackrel{(2.24)}{\subseteq} S_2 \stackrel{(2.23)}{\subseteq} F \quad X = C(1, T'); |T'| > 2. \end{array} \right.$$

This means that for types $X \neq A(\ell, \ell), C(1, 2), C(T, 2), BC(1, 1)$, $F - F \subseteq F$ and so F is a subgroup of A^0 as $0 \in F$. Also we get using (2.23), (2.21) and (2.24) that

$$F = \begin{cases} S_i & \text{if } X \neq C(1, T), BC(T, T'), B(T, T'), B(1, T), B(T, 1); 1 \leq i \leq n, \\ L_i & \text{if } X \neq C(T, T'), C(1, T') \text{ and } (\dot{R}_{re}^i)_{lg} \neq \emptyset; 1 \leq i \leq n, \\ E_i & \text{if } X \neq BC(T, T') \text{ and } (\dot{R}_{re}^i)_{ex} \neq \emptyset; 1 \leq i \leq n, \\ S_2 & \text{if } X = C(1, T'). \end{cases}$$

Finally for types $B(1, T), B(T, 1)$, (2.6) together with (2.22) implies that

$$F + 2S_2 = F + 2S_1 = S_{-\epsilon_0 + \epsilon_t} - 2 \frac{(\epsilon_0, -\epsilon_0 + \epsilon_t)}{(\epsilon_0, \epsilon_0)} S_{\epsilon_0} \subseteq S_{\epsilon_0 + \epsilon_t} = F; \quad (t \in T),$$

and for types $BC(T, T')$ ($|T'| \geq 2$) and $B(T, T')$, (2.9)(a) together with (2.22) and (2.20) implies that

$$F + 2S_1 = F + 2S_2 \subseteq F + L_2 \subseteq F,$$

also for other types by (2.23), we have

$$F + 2S_i \subseteq F + S_i + S_i \subseteq F + S_i \subseteq F; \quad (1 \leq i \leq n).$$

This completes the proof of Claim 5. Now we are ready to complete the proof.

(i) Assume that $X \neq A(\ell, \ell), BC(T, T'), C(T, T'), C(1, T')$. If $X \neq B(T, T'), B(1, T)$ or $B(T, 1)$, by (2.11), $S := F + S_i$ ($1 \leq i \leq n$) is a subgroup of A^0 and so $F + S \subseteq S$. Now if $X = B(T, T'), B(1, T), B(T, 1)$, by (2.23) and (2.22), $S := S_1 = S_2$ and $F + S \subseteq S$. This together with (2.10), (2.11), (2.8) and Proposition 2.1 completes the proof.

(ii) Let $X = BC(1, T), BC(T, T')$ with $|T|, |T'| > 1$. Then $S := S_1 = S_2$ by (2.22) and so by (2.23), $F + S \subseteq S$. Also for $i \in \{1, 2\}$, by (2.20), $F + E_i \subseteq F$ and by (2.11), $F = L_i$ if $(\dot{R}_{re}^i)_{lg} \neq \emptyset$. Therefore we have $2F + E_i \subseteq E_i$ if $(\dot{R}_{re}^i)_{lg} \neq \emptyset$.

Finally by (2.9), we have $S_i + E_i \subseteq S_i$ and $E_i + 4S_i \subseteq E_i$. Now we are done using (2.8), (2.10), (2.16) and proposition 2.1.

(iii) Let $X = C(1, T')$ with $|T'| > 2$. Taking $S := S_1$, we have $F + S \subseteq F$ by (2.23). Also by (2.11), $F = S_2$, so we are done using (2.9), (2.8), (2.16) and Proposition 2.1.

(iv) Let $X = C(T, T')$ with $|T|, |T'| > 2$. Using (2.11), we have $F = S_1 = S_2$ and so we are done using (2.9) together with (2.8), (2.16) and Proposition 2.1. \square

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